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Theory of an Accurate Intermediary Orbit for Satellite Astronomy[1]

John P. Vinti

(May 16, 1961)

This paper derives an accurate intermediary orbit of an artificial satellite of an oblate planet. The drag-free motion takes place under the action of a gravitational potential which fits the even zonal harmonics exactly through the second and approximately through the fourth, in the case of the earth. This potential leads to separability of the Hamilton-Jacobi equation.

Two alternative sets of orbital elements are set forth. The first set is related directly to initial conditions, but requires numerical factoring of a certain quartic to evaluate some of the integrals. The second set, on the other hand, permits exact factoring of both quartics that appear, but is not related directly to initial conditions, so that its members can best be obtained by a least-square fit of the solution over many orbital revolutions.

The final solution is given in terms of certain uniformizing variables, whose periodic terms are correct through the second order in the oblateness parameter and whose secular terms are exact, for the intermediary orbit. These exact solutions for the secular terms are expressed by means of certain rapidly converging series, with complete avoidance of elliptic integrals of the third kind. Sections 9 and 10 give a summary and a list of symbols.

1. Introduction

The author has introduced a gravitational potential [1][2] for an axially symmetric planet which accounts rather accurately for its oblateness and yet still leads to separability of the problem of satellite motion. The resulting solution is expected to yield an "intermediary orbit" somewhat more accurate than those heretofore used, in that it accounts for all of the second zonal harmonic and for more than half of the fourth zonal harmonic. (Previous intermediary orbits [2, 3] have accounted for only part of the second harmonic and have neglected the fourth harmonic.) The residual fourth harmonic, the odd harmonics, the tesseral harmonics, the lunar-solar forces, and aerodynamic and electromagnetic drag are then to be considered as producing perturbations of this intermediary orbit.

Izsak [4] has already given an analytical solution for this intermediary orbit, with both periodic and secular terms correct through the second order in a certain oblateness parameter. His solution makes rather heavy demands on the reader's knowledge of linear fractional transformations and the theory of elliptic functions in the complex plane. The present paper avoids these complications, with elliptic integrals occurring only in the simple forms of the complete first and second kinds. Furthermore the resulting solution not only gives the periodic terms correctly to the second order, but gives the secular terms "exactly"; i.e., to arbitrarily high order. I wish to acknowledge very explicitly, however, that I am greatly indebted to Izsak for the introduction of one of the sets of orbital elements that I use. Knowledge of this set, which permits exact factoring of a certain refractory quartic, has influenced my treatment of the whole problem.

2. Statement of Problem

If ρ, η, ϕ are the oblate spheroidal coordinates introduced in [1] and if r, θ, ϕ and X, Y, Z are the corresponding spherical and rectangular coordinates, then

$$X+iY=r\cos\theta\exp i\phi=[(\rho^2+c^2)(1-\eta^2)]^{\frac{1}{2}}\exp i\phi, \tag{2.1}$$

$$Z=r\sin\theta=\rho\eta, \qquad (-1\leqq\eta\leqq 1). \tag{2.2}$$

[1] This work was supported by the U.S. Air Force, through the Office of Scientific Research of the Air Research and Development Command.

[2] Figures in brackets indicate the literature references at the end of the paper.

Here r is the geocentric distance to the satellite and θ and ϕ are respectively its geocentric declination and right ascension. For sufficiently large r, $\rho \approx r$ and $\eta \approx \sin\theta$.

With the origin taken at the planet's center of mass, the intermediary orbit is then the path of a particle in the approximate potential field

$$V_a = -\mu\rho(\rho^2+c^2\eta^2)^{-\frac{1}{2}}, \tag{2.3}$$

where μ is the product of the gravitational constant and the planetary mass and where

$$c^2 = r_e^2 J_2. \tag{2.4}$$

Here r_e is the equatorial radius of the planet and J_2 is the coefficient of the second zonal harmonic in the expansion of the planet's true potential in spherical harmonics. For the earth $J_2 = (1.08)10^{-3}$, to three significant figures.

According to [1], the coordinates ρ, η, ϕ satisfy the following equations, involving quadratures:

$$t+\beta_1 = \pm\int_{\rho_1}^{\rho} \rho^2 F^{-\frac{1}{2}}(\rho)d\rho \pm c^2\int_0^{\eta} \eta^2 G^{-\frac{1}{2}}(\eta)d\eta, \tag{2.5}$$

$$\beta_2 = \mp\alpha_2\int_{\rho_1}^{\rho} F^{-\frac{1}{2}}(\rho)d\rho \pm \alpha_2\int_0^{\eta} G^{-\frac{1}{2}}(\eta)d\eta, \tag{2.6}$$

$$\phi-\beta_3 = \mp c^2\alpha_3\int_{\rho_1}^{\rho} (\rho^2+c^2)^{-1}F^{-\frac{1}{2}}(\rho)d\rho \pm \alpha_3\int_0^{\eta} (1-\eta^2)^{-1}G^{-\frac{1}{2}}(\eta)d\eta. \tag{2.7}$$

Here $F(\rho)$ and $G(\eta)$ are the quartic polynomials

$$F(\rho) = c^2\alpha_3^2 + (\rho^2+c^2)(-\alpha_2^2+2\mu\rho+2\alpha_1\rho^2), \tag{2.8}$$

$$G(\eta) = -\alpha_3^2 + (1-\eta^2)(\alpha_2^2+2\alpha_1c^2\eta^2). \tag{2.9}$$

The α's and β's are the Jacobi constants, with the energy $\alpha_1 < 0$ for satellite motion and with the polar component of angular momentum $\alpha_3 \gtrless 0$ according as the orbit is direct or retrograde. To orient oneself, note that in the limiting case $c \to 0$ of Keplerian motion the separation constant α_2 reduces to the total angular momentum, $-\beta_1$ to the time of passage through perigee, β_2 to the argument ω of perigee, and β_3 to the right ascension Ω of the ascending node. The Jacobi constants may be determined, at least in principle, from the initial conditions; we have more to say about this point later. The constant ρ_1 is the next-to-the-largest real zero of $F(\rho)$ and thus is that zero of $F(\rho)$ which is closest to the smaller zero of

$$f(\rho) \equiv -\alpha_2^2+2\mu\rho+2\alpha_1\rho^2. \tag{2.10}$$

To solve (2.5) through (2.9) for ρ, η, and ϕ as functions of t, we must first solve (2.5) and (2.6) for ρ and η and then substitute the results $\rho(t)$ and $\eta(t)$ into (2.7) to determine $\phi(t)$. To do so we must first evaluate the above six integrals, which we shall obtain in terms of certain uniformizing variables. In turn, evaluating these integrals presupposes knowing how to factor the quartics $F(\rho)$ and $G(\eta)$.

3. Factoring the Quartics: Orbital Elements a_0, e_0, i_0, β_1, β_2, β_3

In the case of elliptic motion ($c=0$) the perigee and apogee radii r_1 and r_2 would be the two zeros of $f(\rho)$, viz,

$$r_1 = a_0(1-e_0), \tag{3.1}$$

$$r_2 = a_0(1+e_0), \tag{3.2}$$

where

$$a_0 \equiv -\frac{1}{2}\,\mu\alpha_1^{-1} \tag{3.3}$$

$$\epsilon_0^2 \equiv 1 + 2\alpha_1\alpha_2^2\mu^{-2}. \tag{3.4}$$

In our present problem, with $c \neq 0$, we may still *define* constants a_0 and ϵ_0 by (3.3) and (3.4), as well as another constant

$$i_0 \equiv \cos^{-1}(\alpha_3/\alpha_2). \tag{3.5}$$

The constants a_0, ϵ_0, i_0, β_1, β_2, and β_3 constitute *one possible set of orbital constants*. We may also introduce the corresponding semi-latus rectum p_0, defined by

$$p_0 \equiv a_0(1-\epsilon_0^2), \tag{3.6}$$

so that

$$\alpha_2^2 = \mu p_0. \tag{3.7}$$

A determination of α_1, α_2, and α_3 would then furnish a_0, ϵ_0, and i_0. If the subscript i denotes initial values, then (per unit mass)

$$\alpha_1 = \tfrac{1}{2}u_i^2 - \mu\rho_i(\rho_i^2 + c^2\eta_i^2)^{-1}, \tag{3.8}$$

$$\alpha_3 = r_i^2\cos^2\theta_i\dot{\phi}_i = X_i\dot{Y}_i - Y_i\dot{X}_i, \tag{3.9}$$

where u means speed and a superscript dot denotes the time derivative. Also, by equations (50), (59.1), (13.2), (10.2), and (55) of [1],

$$\alpha_2^2 = (1-\eta_i^2)^{-1}[(\rho_i^2 + c^2\eta_i^2)^2\dot{\eta}_i^2 + \alpha_3^2 - 2\alpha_1 c^2\eta_i^2(1-\eta_i^2)]. \tag{3.10}$$

Thus a knowledge of the initial coordinates and their initial derivatives would determine the α's and thus the constants a_0, ϵ_0, and i_0.

A knowledge of their numerical values would then permit a numerical solution of the quartic equation $F(\rho) = 0$ and thus furnish the numerical values of ρ_1, ρ_2, A, and B necessary to factor $F(\rho)$ into the form

$$F(\rho) \equiv -2\alpha_1(\rho-\rho_1)(\rho_2-\rho)(\rho^2 + A\rho + B), \tag{3.11}$$

where ρ_1 and ρ_2 are the zeros of $F(\rho)$ closest to the values r_1 and r_2. Then, in the intermediary orbit, ρ is restricted to the interval $\rho_1 \leqq \rho \leqq \rho_2$ between two spheroids.

By equating the coefficients of corresponding powers of ρ in (3.11) and (2.8) we find

$$\rho^3: \qquad \rho_1 + \rho_2 - A = -\mu\alpha_1^{-1} = 2a_0, \tag{3.12}$$

$$\rho^2: \qquad B + \rho_1\rho_2 - (\rho_1+\rho_2)A = c^2 - \tfrac{1}{2}\alpha_2^2\alpha_1^{-1} = c^2 + a_0p_0, \tag{3.13}$$

$$\rho: \qquad (\rho_1+\rho_2)B - \rho_1\rho_2 A = -\mu c^2\alpha_1^{-1} = 2a_0c^2, \tag{3.14}$$

$$\rho^0: \qquad \rho_1\rho_2 B = -\tfrac{1}{2}c^2(\alpha_2^2 - \alpha_3^2)\alpha_1^{-1} = a_0p_0c^2\sin^2 i_0, \tag{3.15}$$

with use of (3.3) through (3.7).

By beginning with the zero-order solution $A = B = 0$, $\rho_1 + \rho_2 = r_1 + r_2 = 2a_0$, $\rho_1\rho_2 = r_1r_2 = a_0p_0$, one can solve this set of equations for the four unknowns A, B, $\rho_1+\rho_2$, and $\rho_1\rho_2$, by a method of successive approximations. If

$$k_0 \equiv c^2/p_0^2 \equiv (r_e/p_0)^2 J_2, \tag{3.16}$$

$$x \equiv (1-\epsilon_0^2)^{\frac{1}{2}}, \tag{3.17}$$

$$y \equiv \alpha_3/\alpha_2 = \cos i_0, \tag{3.18}$$

the second-order solution, through k_0^2, is

$$A=-2k_0p_0y^2[1+k_0(2x^2-3x^2y^2-4+8y^2)+\ \ldots],\tag{3.19}$$

$$B=k_0p_0^2(1-y^2)[1+k_0(4y^2-x^2y^2)+\ \ldots],\tag{3.20}$$

$$\rho_1+\rho_2=2p_0x^{-2}[1-k_0x^2y^2-k_0^2x^2y^2(2x^2-3x^2y^2-4+8y^2)+\ \ldots],\tag{3.21}$$

$$\rho_1\rho_2=p_0^2x^{-2}[1+k_0y^2(x^2-4)-k_0^2y^2(12x^2-x^4-20x^2y^2-16+32y^2+x^4y^2)+\ \ldots].\tag{3.22}$$

The constants

$$a\equiv\tfrac{1}{2}(\rho_1+\rho_2),\tag{3.23}$$

$$e\equiv\frac{\rho_2-\rho_1}{\rho_2+\rho_1},\tag{3.24}$$

$$p\equiv a(1-e^2)\tag{3.25}$$

will occur throughout the evaluation of the ρ-integrals. In terms of a_0, e_0, and i_0, their values, to the second order in k_0, then satisfy

$$p/p_0=1+2k_0y^2(x^2-2)+k_0^2y^2(3x^4-2x^4y^2-16x^2+24x^2y^2+16-32y^2)+\ \ldots,\tag{3.26}$$

$$p_0/p=1+2k_0y^2(2-x^2)+k_0^2y^2(-3x^4+6x^4y^2+16x^2-40x^2y^2-16+48y^2)+\ \ldots,\tag{3.27}$$

$$\frac{1-e^2}{1-e_0^2}=1+k_0y^2(3x^2-4)+k_0^2y^2(5x^4-2x^4y^2-20x^2+28x^2y^2+16-32y^2)+\ \ldots,\tag{3.28}$$

$$\left(\frac{1-e^2}{1-e_0^2}\right)^{\frac{1}{2}}=1+\frac{1}{2}\,k_0y^2(3x^2-4)+\frac{1}{8}\,k_0^2y^2(20x^4-17x^4y^2-80x^2+136x^2y^2+64-144y^2)+\ \ldots,\tag{3.29}$$

$$(1-e^2)^{\frac{1}{2}}p^{-1}=a_0^{-1}(1-e_0^2)^{-\frac{1}{2}}[1+\tfrac{1}{2}\,k_0y^2(4-x^2)+\tfrac{1}{8}\,k_0^2y^2(-4x^4+7x^4y^2+48x^2$$
$$-104x^2y^2-64+176y^2)+\ \ldots].\tag{3.30}$$

In those places where e occurs alone in the theory, i.e., not in the combination $1-e^2$, it may of course be found by use of

$$e\equiv[1-(1-e^2)]^{\frac{1}{2}}.\tag{3.31}$$

By (3.28) this results in

$$e=[e_0^2+k_0x^2y^2(4-3x^2)+\ \ldots]^{\frac{1}{2}},\tag{3.32}$$

so that when e_0 is comparable to k_0 it is not feasible to expand e in a power series in k_0; indeed if $e_0=0$, we should need a power series in $k_0^{\frac{1}{2}}$.

Direct use of this second-order solution in factoring $F(\rho)$ will lead to ρ-integrals that have secular terms correct only to $O(k_0^2)$. Since we are aiming at arbitrarily high accuracy for the secular terms, we include it here for other purposes. The most important of these purposes is to furnish information about the orders of various quantities in k_0; e.g., A and B are both of order k_0. Such information will be necessary in carrying out the solution of (2.5) through (2.7) for the periodic terms. The second purpose is to furnish a convenient starting point for any investigators who may choose to use a_0, e_0, and i_0, along with the β's, as orbital elements, and who will therefore have to solve the equation $F(\rho)=0$ numerically. A third purpose is for use in calculating the mean motions to first order, for comparison with other theories.

The quartic $G(\eta)$, which is quadratic in η^2, may be factored either as

$$G(\eta)\equiv-2\alpha_1c^2(\eta_0^2-\eta^2)(\eta_2^2-\eta^2)\tag{3.33}$$

or as

$$G(\eta)\equiv(\alpha_2^2-\alpha_3^2)\eta^4(\eta^{-2}-\eta_0^{-2})(\eta^{-2}-\eta_2^{-2}). \tag{3.34}$$

If we use the latter form, we find on comparing (3.34) with (2.9) that η_0^{-2} and η_2^{-2} are the roots of the following quadratic equation in η^{-2}:

$$(\alpha_2^2-\alpha_3^2)\eta^{-4}+(2\alpha_1c^2-\alpha_2^2)\eta^{-2}-2\alpha_1c^2=0. \tag{3.35}$$

Thus

$$\eta_0^{-2}=\tfrac{1}{2}(\alpha_2^2-2\alpha_1c^2)(\alpha_2^2-\alpha_3^2)^{-1}(1+W^{\frac{1}{2}}), \tag{3.36}$$

$$\eta_2^{-2}=\tfrac{1}{2}(\alpha_2^2-2\alpha_1c^2)(\alpha_2^2-\alpha_3^2)^{-1}(1-W^{\frac{1}{2}}), \tag{3.37}$$

where

$$W\equiv 1+8\alpha_1c^2(\alpha_2^2-\alpha_3^2)(\alpha_2^2-2\alpha_1c^2)^{-2}. \tag{3.38}$$

From (3.36) and (3.38) it follows that for $\alpha_1<0$

$$\eta_0^2\leqq\alpha_2^{-2}(\alpha_2^2-\alpha_3^2)\leqq 1 \tag{3.39}$$

and we shall see below that for satellite motion $\eta_2^2\gg 1$. Since η^2 cannot exceed 1, it follows that in the actual motion η always lies in the interval $-\eta_0\leqq\eta\leqq\eta_0$ between two hyperboloids.

We readily find that

$$\eta_0=(\sin i_0)\,[1-\tfrac{1}{2}k_0x^2y^2+\tfrac{1}{8}k_0^2x^4y^2(7y^2-4)+\ldots], \tag{3.40}$$

$$(1-\eta_0^2)^{-\frac{1}{2}}=|\sec i_0|[1-\tfrac{1}{2}k_0x^2(1-y^2)+\tfrac{1}{8}k_0^2x^4(1-y^2)(5y^2-1)+\ldots], \tag{3.41}$$

$$\eta_2^{-2}=k_0x^2(1-k_0x^2y^2+\ldots), \tag{3.42}$$

$$(\eta_0/\eta_2)^2=k_0x^2(\sin^2 i_0)(1-2k_0x^2y^2+\ldots). \tag{3.43}$$

Note that $\eta_2^{-2}\leqq k_0\approx 10^{-3}$, so that $\eta_2^2\geqq 1000$.

4. Factoring the Quartics: Orbital Elements a, e, I, β_1, β_2, β_3

If we equate the coefficients of powers of η^2 in (3.33) with those of corresponding powers in (2.9), we find

$$\eta_0^2+\eta_2^2=1-\frac{\alpha_2^2}{2\alpha_1c^2}=1+a_0p_0/c^2, \tag{4.1}$$

$$\eta_0^2\eta_2^2=-\frac{\alpha_2^2-\alpha_3^2}{2\alpha_1c^2}=\frac{a_0p_0}{c^2}\sin^2 i_0. \tag{4.2}$$

If in (3.12) through (3.15) we use (3.23) through (3.25), we find

$$2a-A=2a_0, \tag{4.3}$$

$$B+ap-2Aa=c^2+a_0p_0, \tag{4.4}$$

$$2aB-Aap=2a_1c^2, \tag{4.5}$$

$$Bap=a_0p_1c^2\sin^2 i_0. \tag{4.6}$$

Suppose we now regard a, e, and

$$\eta_0\equiv\sin I \tag{4.7}$$

as known. (When we adopt a, e, I, and the β's as orbital elements we are certainly assuming so; we discuss later how they may be determined.) Then in treating the ρ-integrals we have five unknowns, viz, a_0, e_0, i_0, A, and B, and in treating the η-integrals one additional unknown, viz, η_2. Altogether then, we have six unknowns and we have six equations with which to determine them, (4.1) through (4.6). With *these* orbital elements, viz, a, e, and η_0, however, it

turns out that the system can be solved *exactly* and with considerable ease, for the required unknowns. This property of a, ϵ, and η_0 was first pointed out by Izsak [4].

To carry out the solution, first eliminate η_2 from (4.1) and (4.2). The result is

$$\frac{\alpha_2^2-\alpha_3^2}{\alpha_2^2}\equiv\sin^2 i_0=\eta_0^2+\frac{c^2\eta_0^2(1-\eta_0^2)}{a_0p_0}. \tag{4.8}$$

On inserting (4.8) into (4.6), we find

$$Bap=c^2\eta_0^2\,[a_0p_0+c^2(1-\eta_0^2)]. \tag{4.9}$$

If we now use (4.3) to eliminate a_0 from (4.5) and (4.9) to eliminate a_0p_0 from (4.4), we find a pair of simultaneous linear equations for A and B:

$$(ap-c^2)A-2aB=-2ac^2, \tag{4.10}$$

$$2c^2\eta_0^2aA+(ap-c^2\eta_0^2)\,B=c^2\eta_0^2(ap-c^2\eta_0^2). \tag{4.11}$$

Their solution is

$$A=-\frac{2ac^2(1-\eta_0^2)(ap-c^2\eta_0^2)}{(ap-c^2)(ap-c^2\eta_0^2)+4a^2c^2\eta_0^2}, \tag{4.12}$$

$$B=c^2\eta_0^2\,\frac{(ap-c^2)(ap-c^2\eta_0^2)+4a^2c^2}{(ap-c^2)\,(ap-c^2\eta_0^2)+4a^2c^2\eta_0^2}. \tag{4.13}$$

Then, from (4.3) and (4.12)

$$-\frac{\mu}{2a\alpha_1}\equiv\frac{a_0}{a}=1-\frac{A}{2a}=1+\frac{c^2(1-\eta_0^2)(ap-c^2\eta_0^2)}{(ap-c^2)\,(ap-c^2\eta_0^2)+4a^2c^2\eta_0^2} \tag{4.14}$$

and from (4.9) and (4.13)

$$-\frac{\alpha_2^2}{2\alpha_1}\equiv a_0p_0=-c^2(1-\eta_0^2)+ap\,\frac{(ap-c^2)(ap-c^2\eta_0^2)+4a^2c^2}{(ap-c^2)\,(ap-c^2\eta_0^2)+4a^2c^2\eta_0^2}. \tag{4.15}$$

Equation (4.8) and the relation $\eta_0\equiv\sin I$ give

$$\alpha_3=\alpha_2\left(1-\frac{c^2\eta_0^2}{a_0p_0}\right)^{\frac{1}{2}}\cos I, \tag{4.15a}$$

so that (4.15) and (4.15a) determine α_3. Finally, to obtain η_2, combine (4.2) and (4.6) to obtain $\eta_2^{-2}=c^4\eta_0^2\,(Bap)^{-1}$ and then use (4.13). The result is

$$\eta_2^{-2}=\frac{c^2}{ap}\,\frac{(ap-c^2)\,(ap-c^2\eta_0^2)+4a^2c^2\eta_0^2}{(ap-c^2)(ap-c^2\eta_0^2)+4a^2c^2}. \tag{4.16}$$

This completes the solution for the required unknowns when the orbital elements are a, ϵ, and I. In terms of these orbital elements we can now factor the two quartics $F(\rho)$ and $G(\eta)$ exactly and thus evaluate all the integrals.

With use of the old oblateness parameter $k_0\equiv c^2/p_0^2$ and a new one, suitable for use with this second set of orbital elements, viz,

$$k\equiv c^2/p^2\equiv(r_e/p)^2J_2, \tag{4.17}$$

we can readily show that, to the first order, the equations of sections 3 and 4 give similar results. Thus we readily obtain

$$A\approx-2k_0p_0\cos^2 i_0\approx-2kp\cos^2 I, \tag{4.18}$$

$$B\approx k_0p_0^2\sin^2 i_0\approx kp^2\sin^2 I, \tag{4.19}$$

$$\frac{a_0}{a}\approx1+k_0(1-e_0^2)\cos^2 i_0\approx1+k(1-e^2)\cos^2 I, \tag{4.20}$$

$$\frac{p_0}{p}\approx 1+2k_0(1+\epsilon_0^2)\cos^2 i_0\approx 1+2k(1+\epsilon^2)\cos^2 I, \tag{4.21}$$

$$\frac{1-\epsilon_0^2}{1-\epsilon^2}\approx 1+k_0(1+3\epsilon_0^2)\cos^2 i_0\approx 1+k(1+3\epsilon^2)\cos^2 I, \tag{4.22}$$

$$\frac{\sin i_0}{\sin I}\approx 1+\frac{1}{2}k_0(1-\epsilon_0^2)\cos^2 i_0\approx 1+\frac{1}{2}k(1-\epsilon^2)\cos^2 I, \tag{4.23}$$

$$\eta_2^{-2}\approx k_0(1-\epsilon_0^2)+\ \ldots\ \approx k(1-\epsilon^2)+\ \ldots, \tag{4.24}$$

$$(\eta_0/\eta_2)^2\approx k_0(1-\epsilon_0^2)\sin^2 i_0\approx k(1-\epsilon^2)\sin^2 I+\ \ldots\ . \tag{4.25}$$

With either set of orbital elements, the final solution will be given in terms of a, ϵ, I, β_1, β_2, and β_3, and certain angles E, v, and ψ, analogous respectively to the eccentric anomaly, the true anomaly, and the argument of latitude in elliptic motion. Once one knows a, ϵ, and I, one can then determine the β's by observing, at various times, whatever quantities will best serve to determine E, v, and ψ.

If one is using a_0, ϵ_0, and i_0 as elements, one has to determine them from initial values or from some procedure equivalent to determining initial values. One then has to factor the quartic $F(\rho)$ numerically to determine a, ϵ, and $\eta_0\equiv\sin I$.

If one is using a, ϵ, and I as elements, one has to determine them by following the orbit for many revolutions and then applying some sort of iterated least-square process. In this case one can then find the Jacobi constants α_1, α_2, and α_3 and thus a_0, ϵ_0, and i_0 by means of the equations of this section.

In any event, the determination of orbital elements, by comparison of theory and observation, is ordinarily considered a completely separate problem in celestial mechanics from the calculation of the motion for given orbital elements. We have included the above remarks only to aid in the possible application of the present theoretical solution. Indeed the problem is further complicated by the small perturbations that occur in practice.

In the rest of the paper we simply assume a, ϵ, $\eta_0\equiv\sin I$, β_1, β_2, and β_3 to be known and then complete the solution for the intermediary orbit. There will be one restriction, however. The method of evaluating the ρ-integrals will be found to depend on the orbital inclination i_0 or I. For values of I less than $1°54'$ or greater than $178°6'$, a different approach would be needed. We shall therefore restrict considerations in the present paper to orbits that have inclinations between these two values, thereby ruling out equatorial or almost equatorial orbits.

5. The ρ-Integrals

In (2.5) through (2.7) the ρ-integrals are

$$R_1\equiv\pm\int_{\rho_1}^{\rho}\rho^2F^{-\frac{1}{2}}(\rho)d\rho, \tag{5.1}$$

$$R_2\equiv\pm\int_{\rho_1}^{\rho}F^{-\frac{1}{2}}(\rho)d\rho, \tag{5.2}$$

$$R_3\equiv\pm\int_{\rho_1}^{\rho}(\rho^2+\epsilon^2)^{-1}F^{-\frac{1}{2}}(\rho)d\rho, \tag{5.3}$$

where $F(\rho)$ is given by (3.11), A and B by (4.12) and (4.13), p by (3.25), and ρ_1 and ρ_2 by

$$\rho_1=a(1-\epsilon), \qquad \rho_2=a(1+\epsilon). \tag{5.4}$$

Equations (5.4) follow from (3.23) and (3.24). Then

$$F^{-\frac{1}{2}}(\rho)=(-2\alpha_1)^{-\frac{1}{2}}[(\rho-\rho_1)(\rho_2-\rho)]^{-\frac{1}{2}}\rho^{-1}(1+A\rho^{-1}+B\rho^{-2})^{-\frac{1}{2}}. \tag{5.5}$$

If, for convenience, we put

$$A \equiv -2b_1, \qquad B \equiv b_2^2, \qquad \lambda \equiv b_1/b_2 \tag{5.6}$$

and

$$h \equiv b_2/\rho, \tag{5.7}$$

then

$$(1+A\rho^{-1}+B\rho^{-2})^{-\frac{1}{2}} \equiv (1-2\lambda h+h^2)^{-\frac{1}{2}} \tag{5.8}$$

$$= \sum_{n=0}^{\infty} h^n P_n(\lambda), \tag{5.9}$$

if $|h|<1$ and $|\lambda|<1$.

The expansion (5.9) will be a convenient tool for evaluating all the ρ-integrals, whenever it can be used. To see when, use (5.6), (5.7), (4.18), and (4.19). Then to $O(k)$

$$b_1 = kp \cos^2 I, \qquad b_2 = k^{\frac{1}{2}} p \sin I,$$

so that $b_2/\rho \ll 1$ and

$$\lambda = k^{\frac{1}{2}} \cos^2 I \csc I > 0.$$

Thus to $O(k)$, $|\lambda|<1$ whenever

$$k^{\frac{1}{2}} \cos^2 I \csc I < 1,$$

or whenever

$$\tan^2 I + \tan^4 I > k$$

or

$$2 \tan^2 I > (1+4k)^{\frac{1}{2}} - 1.$$

To $O(k)$, this becomes

$$\tan^2 I > k.$$

But $k=(r_e/p)^2 J_2$ and $J_2 = (1.08)10^{-3}$ for the earth, so that $\lambda<1$ whenever

$$|\tan I| > 0.033\, r_e/p.$$

For close orbits $r_e/p \approx 1$, so that the necessary condition becomes

$$I_c < I < 180^\circ - I_c, \qquad I_c \approx 1^\circ 54'.$$

The expansion will thus work for all satellite orbits that are inclined more than $1^\circ 54'$ to the equator. For those orbits that lie closer to the equator one must use some other method to evaluate the ρ-integrals. On inserting (5.7) and (5.9) into (5.5), we find

$$F(\rho)^{-\frac{1}{2}} = (-2\alpha_1)^{-\frac{1}{2}} \sum_{n=0}^{\infty} b_2^n P_n(\lambda) \rho^{-1-n} [(\rho-\rho_1)(\rho_2-\rho)]^{-\frac{1}{2}}. \tag{5.10}$$

With use of (5.1), (5.10), and (5.6), we find

$$(-2\alpha_1)^{\frac{1}{2}} R_1 = \int_{\rho_1}^{\rho} (\rho+b_1)[(\rho-\rho_1)(\rho_2-\rho)]^{-\frac{1}{2}} (\pm d\rho) + \sum_{n=2}^{\infty} b_2^n P_n(\lambda) \int_{\rho_1}^{\rho} \rho^{1-n} [(\rho-\rho_1)(\rho_2-\rho)]^{-\frac{1}{2}} (\pm d\rho). \tag{5.11}$$

Each of the separate integrals in (5.11) is a multiple-valued function of ρ. It is appropriate to change variables in each to a uniformizing variable E or v, analogous respectively to the eccentric and true anomalies in elliptic motion. We define E and v by requiring them to satisfy

$$\rho = a(1-e\cos E) = (1+e\cos v)^{-1} p \tag{5.12}$$

and always to increase with time. Then, from (5.12), (5.4), and (3.23) through (3.25), we find

$$[(\rho-\rho_1)(\rho_2-\rho)]^{-\frac{1}{2}} (\pm d\rho) = dE = (1-e^2)^{\frac{1}{2}} (1+e\cos v)^{-1} dv \tag{5.13}$$

5.1. The Integral R_1

On introducing E into the first integral in (5.11) and v into each of the integrals in the series, we find

$$(-2\alpha_1)^{\frac{1}{2}}R_1=b_1E+a(E-e\sin E)+(1-e^2)^{\frac{1}{2}}p\sum_{n=2}^{\infty}(b_2/p)^nP_n(\lambda)\int_0^v(1+e\cos v)^{n-2}dv. \quad (5.14)$$

To investigate the convergence of the series we write

$$S_1\equiv\sum_{n=2}^{\infty}(b_2/p)^nP_n(\lambda)\int_0^v(1+e\cos v)^{n-2}dv \quad (5.15)$$

$$=(b_2/p)^2\sum_{m=0}^{\infty}(b_2/p)^mP_{m+2}(\lambda)\int_0^v(1+e\cos v)^mdv. \quad (1.16)$$

Then

$$|S_1|\leqq(b_2/p)^2\sum_{m=0}^{\infty}(b_2/p)^m|P_{m+2}(\lambda)|(1+e)^mv \quad (5.17)$$

and since $|P_n(\lambda)|\leqq 1$ for all $|\lambda|\leqq 1$, we have

$$|S_1|\leqq(b_2/p)^2v\sum_{m=0}^{\infty}\left[\frac{b_2(1+e)}{p}\right]^m \quad (5.18)$$

$$\leqq\frac{(b_2/p)^2v}{1-b_2(1+e)/p}. \quad (5.19)$$

By (4.19) and (5.6), however, $b_2/p\approx k^{\frac{1}{2}}\sin I$ and of course $1+e\leqq 2$, so that

$$b_2(1+e)/p\leqq 2k^{\frac{1}{2}}\sin I\leqq 2k^{\frac{1}{2}}. \quad (5.20)$$

Since $k\approx 10^{-3}$, we have $b_2(1+e)/p\leqq 0.063$. Thus the series S_1 converges, and converges more rapidly than a geometric series of common ratio $\approx 1/16$. Actually, since we have shown that the series S_1 converges absolutely, we can regroup it into the sum of a series S_{1e} containing only the even values of n and a series S_{1o} containing only the odd values of n. It is then a simple matter to show that $S_1\equiv S_{1e}+S_{1o}$ converges more rapidly than the geometric series $[1+b_2(1+e)/p]\cdot\sum_{n=0}^{\infty}[b_2(1+e)/p]^{2n}$. That is, we can actually expect the convergence of S_1 to be as rapid as that of a geometric series of common ratio $[b_2(1+e)/p]\leqq 4k\approx 1/250$.

To decompose the series S_1 into a part proportional to v and a periodic part, note first that if

$$f_m(v)\equiv\int_0^v(1+e\cos v)^mdv, \quad (5.21)$$

then $f_n(v)-(2\pi)^{-1}f_m(2\pi)v$ is an odd function of v, of period 2π. But $f_m(2\pi)=2f_m(\pi)$, so that we obtain

$$f_m(v)\equiv\int_0^v(1+e\cos v)^mdv=\pi^{-1}v\int_0^{\pi}(1+e\cos v)^mdv+\sum_{j=1}^{\infty}c_{mj}\sin jv. \quad (5.22)$$

To obtain, to any order in k, the parts of the ρ-integrals proportional to v, we shall need to consider all integral values of m in $\pi^{-1}f_m(\pi)$. To obtain the periodic parts correct to order k^2, we shall need values of m only up to 4.

To obtain a convenient expression for the v-term of $f_m(v)$, note that [5]

$$\int_0^{\pi}(z+\sqrt{z^2-1}\cos v)^mdv=\pi P_m(z), \quad (5.23)$$

for all values of z, including real values greater than unity. Here $P_m(z)$ is the Legendre polynomial

$$P_m(z)\equiv\frac{2^{-m}}{m!}\frac{d^m}{dz^m}(z^2-1)^m, \quad (5.24)$$

of the same polynomial form in z that holds when $|z|<1$, when it can be defined by the usual generating function. If we put

$$z=(1-e^2)^{-\frac{1}{2}} \tag{5.25}$$

in (5.23), we find

$$\int_0^\pi (1+e\cos v)^m dv=\pi(1-e^2)^{m/2}P_m[(1-e^2)^{-\frac{1}{2}}] \tag{5.26}$$

$$=\pi R_m(\sqrt{1-e^2}), \tag{5.27}$$

where

$$R_m(x)\equiv x^m P_m(1/x) \qquad (0\leqq x\leqq 1). \tag{5.28}$$

Thus $R_m(x)$ is a polynomial of degree $[m/2]$ in x^2. The first few of these polynomials are given in table 1.

TABLE 1

m	$R_m(x)$
0	1
1	1
2	$\frac{1}{2}(3-x^2)$
3	$\frac{1}{2}(5-3x^2)$
4	$(\frac{1}{8})(35-30x^2+3x^4)$
5	$(\frac{1}{8})(63-70x^2+15x^4)$
6	$(\frac{1}{16})(231-315x^2+105x^4-5x^6)$

From (5.22) and (5.27) we then obtain

$$\int_0^v (1+e\cos v)^m dv=R_m(\sqrt{1-e^2})v+\sum_{j=1}^{\infty} c_{mj}\cdot\sin jv. \tag{5.29}$$

Through $m=4$, the coefficients c_{mj} are easy to find, simply by binomial expansion of the integrand and conversion to a trigonometric polynomial.
The results are given in the following table:

TABLE 2. *Coefficients c_{mj} in (5.29)*

m \ j	1	2	3	4
0				
1	1			
2	$2e$	$e^2/4$		
3	$3e+3e^3/4$	$3e^2/4$	$e^3/12$	
4	$4e+3e^3$	$3e^2/2+e^4/4$	$e^3/3$	$e^4/32$

In (5.14), $b_2^n=O(k^{n/2})$ and $P_n(\lambda)=O(\lambda^0)=O(k^0)$ if n is even or $P_n(\lambda)=O(\lambda)=O(k^{\frac{1}{2}})$ if n is odd. On inserting (5.29) into (5.14), using table 2, and keeping periodic terms through $O(k^2)$ only, we find

$$(-2\alpha_1)^{\frac{1}{2}}R_1=b_1E+a(E-e\sin E)+A_1 v+\sum_{j=1}^{2} A_{1j}\cdot\sin jv, \tag{5.30}$$

where

$$A_1=(1-e^2)^{\frac{1}{2}}p\sum_{n=2}^{\infty}(b_2/p)^n P_n(\lambda)R_{n-2}(\sqrt{1-e^2}) \tag{5.31}$$

and

$$A_{11}=\frac{3(1-e^2)^{\frac{1}{2}}}{4p^3}(-2b_1b_2^2p+b_2^4)e, \tag{5.32}$$

$$A_{12}=\frac{3(1-e^2)^{\frac{1}{2}}}{32p^3}b_2^4e^2. \tag{5.33}$$

The above proof of convergence of the series for R_1 also shows the rapid convergence of the series (5.31) for the coefficient A_1 of the v-term.

5.2. The Integral R_2

On inserting (5.10) into (5.2) and using (5.12) and (5.13), we find

$$(-2\alpha_1)^{\frac{1}{2}}R_2=(1-e^2)^{\frac{1}{2}}p^{-1}\sum_{n=0}^{\infty}(b_2/p)^n P_n(\lambda)\int_0^v (1+e\cos v)^n dv. \tag{5.34}$$

As above, one can show at once that this series converges more rapidly than the geometric series $v\sum_{n=0}^{\infty}[b_2(1+e)/p]^n$, with common ratio $b_2(1+e)/p \leqq 2k^{\frac{1}{2}} \approx 0.066$. The same proof then applies to the coefficient A_2 in

$$(-2\alpha_1)^{\frac{1}{2}}R_2=A_2 v+\sum_{j=1}^{4}A_{2j}\sin jv. \tag{5.35}$$

Using the same methods as for R_1, we then find

$$A_2=(1-e^2)^{\frac{1}{2}}p^{-1}\sum_{n=0}^{\infty}(b_2/p)^n P_n(\lambda)R_n(\sqrt{1-e^2}) \tag{5.36}$$

and, through periodic terms of $O(k^2)$:

$$A_{21}=(1-e^2)^{\frac{1}{2}}p^{-1}e[b_1p^{-1}+(3b_1^2-b_2^2)p^{-2}-\tfrac{9}{2}b_1b_2^2(1+e^2/4)p^{-3}+\tfrac{3}{8}b_2^4(4+3e^2)p^{-4}], \tag{5.37}$$

$$A_{22}=(1-e^2)^{\frac{1}{2}}p^{-1}\left[\frac{e^2}{8}(3b_1^2-b_2^2)p^{-2}-\tfrac{9}{8}e^2b_1b_2^2p^{-3}+\frac{3b_2^4}{32}(6e^2+e^4)p^{-4}\right], \tag{5.38}$$

$$A_{23}=(1-e^2)^{\frac{1}{2}}p^{-1}\frac{e^3}{8}(-b_1b_2^2p^{-3}+b_2^4p^{-4}), \tag{5.39}$$

$$A_{24}=(1-e^2)^{\frac{1}{2}}p^{-5}\frac{3}{256}b_2^4e^4. \tag{5.40}$$

5.3. The Integral R_3

If in (5.3) we now use the binomial expansion

$$(\rho^2+c^2)^{-1}=\rho^{-2}\sum_{j=0}^{\infty}(-1)^j c^{2j}\rho^{-2j} \tag{5.41}$$

and insert the expressions (5.10), (5.12), and (5.13), we find

$$(-2\alpha_1)^{\frac{1}{2}}R_3=(1-e^2)^{\frac{1}{2}}p^{-3}\int_0^v\sum_{j=0}^{\infty}(-1)^j(c/p)^{2j}(1+e\cos v)^{2j}\sum_{n=0}^{\infty}(b_2/p)^n P_n(\lambda)(1+e\cos v)^{n+2}dv, \tag{5.42}$$

where the integrand is the product of two series, each of which converges absolutely for any value of v. Then [6] it is equal to the series formed by summing the products of the individual terms, taken in any order, and this resulting series is itself absolutely convergent, for any value of v. It is therefore uniformly convergent, by the Weierstress M-test [7], so that it may be integrated term by term.

Let us now rewrite (5.42) in the form

$$(-2\alpha_1)^{\frac{1}{2}}R_3=(1-e^2)^{\frac{1}{2}}p^{-3}\int_0^v\sum_{m=0}^{\infty}D_m(1+e\cos v)^{m+2}dv. \tag{5.43}$$

Here

$$D_m=\sum d_j\delta_{n'}, \tag{5.44}$$

summed over all those non-negative integral values of j and n' for which

$$2j+n'=m, \tag{5.45}$$

and where

$$d_j=(-1)^j(c/p)^{2j}, \tag{5.46}$$

$$\delta_{n'}=(b_2/p)^{n'}P_{n'}(\lambda). \tag{5.47}$$

Then, because of the uniform convergence,

$$(-2\alpha_1)^{\frac{1}{2}}R_3=(1-e^2)^{\frac{1}{2}}p^{-3}\sum_{m=0}^{\infty}D_m\int_0^v(1+e\cos v)^{m+2}dv. \tag{5.48}$$

It is interesting to investigate here the rapidity of convergence of the series

$$S\equiv\sum_{n=0}^{\infty}D_m\int_0^v(1+e\cos v)^{m+2}dv. \tag{5.49}$$

If m is even, we have $m=2i$ and $n'=2n$, so that $2j+2n=2i$ and $j=i-n$. Then

$$D_m=D_{2i}=\sum_{n=0}^{i}d_{i-n}\delta_{2n}=\sum_{n=0}^{i}(-1)^{i-n}(c/p)^{2i-2n}(b_2/p)^{2n}P_{2n}(\lambda), \tag{5.50}$$

so that

$$|D_{2i}|\leqq(c/p)^{2i}\sum_{n=0}^{i}(b_2/c)^{2n}. \tag{5.51}$$

But $c^2=kp^2$ and $b_2^2\approx kp^2\sin^2 I$, by (4.19), so that

$$(b_2/c)^2\approx\sin^2 I\leqq 1.$$

Then

$$|D_{2i}|\leqq(i+1)k^i. \tag{5.52}$$

If m is odd, we have $m=2i+1$ and $n'=2n+1$, so that

$$D_{2i+1}=\sum_{n=0}^{i}d_{i-n}\delta_{2n+1}=\sum_{n=0}^{i}(-1)^{i-n}(c/p)^{2i-2n}(b_2/p)^{2n+1}P_{2n+1}(\lambda). \tag{5.53}$$

Then

$$|D_{2i+1}|\leqq(c/p)^{2i}(b_2/p)\sum_{n=0}^{i}(b_2/c)^{2n}\leqq k^ik^{\frac{1}{2}}\,(i+1)\sin I. \tag{5.54}$$

Then, breaking up S into an even series S_e and an odd series S_o, we find

$$|S_e|\leqq\sum_{i=0}^{\infty}(i+1)k^i(1+e)^{2i+2}v\leqq(1+e)^2v\sum_{i=0}^{\infty}(i+1)[k(1+e)^2]^i. \tag{5.55}$$

Using $\sum_0^\infty x^i=(1-x)^{-1}$ and $\sum_1^\infty ix^i=x(1-x)^{-2}$, we find $\sum_0^\infty(i+1)x^i=(1-x)^{-2}$, so that

$$|S_e|\leqq\frac{(1+e)^2v}{[1-k(1+e^2)]^2}. \tag{5.56}$$

Similarly

$$|S_o|\leqq k^{\frac{1}{2}}(1+e)^3v\sin I\sum_{i=0}^{\infty}(i+1)[k(1+e)^2]^i \tag{5.57}$$

$$\leqq\frac{k^{\frac{1}{2}}(1+e)^3v\sin I}{[1-k(1+e)^2]^2} \tag{5.58}$$

Thus,

$$|S|\leqq\frac{(1+e)^2v[1+k^{\frac{1}{2}}\,(1+e)\,\sin I]}{[1-k(1+e)^2]^2}. \tag{5.59}$$

The series (5.49) for R_3 thus converges more rapidly than the series expansion of the function given in (5.59).

Through $O(k^2)$ the values of D_m are given by

TABLE 3

m	Order	D_m
0	k^0	$d_0b_0=1$
1	k	$d_0b_1=\bar{b}_1=b_1/p\simeq k\cos^2 I$
2	k	$d_0b_2+d_1b_0=(b_2/p)^2P_2(b_1/b_2)-(c/p)^2$
3	k^2	$d_0b_3+d_1b_1=(b_2/p)^2P_3(b_1/b_2)-(c/p)^2(b_1/p)$
4	k^2	$d_0b_4+d_1b_2+d_2b_0=(b_2/p)^4P_4(b_1/b_2)-(c/p)^2(b_2/p)^2P_2(b_1/b_2)+(c/p)^4$

As with R_1 and R_2, we find

$$(-2\alpha_1)^{\frac{1}{2}}R_3=A_3v+\sum_{j=1}^{\infty}A_{3j}\sin jv, \tag{5.60}$$

where

$$A_3=(1-e^2)^{\frac{1}{2}}p^{-3}\sum_{m=0}^{\infty}D_mR_{m+2}(\sqrt{1-e^2}). \tag{5.61}$$

The remarks about the convergence of the series (5.49) apply also to (5.61).

Since R_3 is multiplied by $c^2=kp^2$ in (2.7), we need periodic terms only through $O(k)$, in order to have periodic terms in the final solution correct through $O(k^2)$. By (5.43) and tables 2 and 3, their coefficients are

$$A_{31}=(1-e^2)^{\frac{1}{2}}p^{-3}e\left[2+b_1p^{-1}\left(3+\frac{3}{4}e^2\right)-p^{-2}\left(\frac{b_2^2}{2}+c^2\right)(4+3e^2)\right], \tag{5.62}$$

$$A_{32}=(1-e^2)^{\frac{1}{2}}p^{-3}\left[\frac{e^2}{4}+\frac{3}{4}\frac{b_1}{p}e^2-p^2\frac{e^4}{4}+\frac{3}{2}e^2\left(\frac{b_2^2}{2}+c^2\right)\right], \tag{5.63}$$

$$A_{33}=(1-e^2)^{\frac{1}{2}}p^{-3}\left[\frac{e^3b^1}{12p}-\frac{e^3}{3p^2}\left(\frac{b_2^2}{2}+c^2\right)\right], \tag{5.64}$$

$$A_{34}=(1-e^2)^{\frac{1}{2}}p^{-3}\left[-\frac{e^4}{32p^2}\left(\frac{b_2^2}{2}+c^2\right)\right]. \tag{5.65}$$

6. The η-Integrals

In (2.5) through (2.7) the η-integrals are

$$N_1=\pm\int_0^{\eta}\eta^2G^{-\frac{1}{2}}(\eta)d\eta, \tag{6.1}$$

$$N_2=\pm\int_0^{\eta}G^{-\frac{1}{2}}(\eta)d\eta, \tag{6.2}$$

$$N_3=\pm\int_0^{\eta}(1-\eta^2)^{-1}G(\eta)^{-\frac{1}{2}}d\eta, \tag{6.3}$$

where $G(\eta)$ is given by (3.34), η_2 by (4.16), α_2^2 by (4.14) and (4.15), and $\alpha_3^2-\alpha_2^2$ finally by (4.8).

In evaluating N_1 and N_2 it is convenient to put

$$\eta=\eta_0\sin\psi, \tag{6.4}$$

where ψ is to be an angle that always increases with time. In the limiting case $c=0$ we should have $\sin\psi=\sin\theta/\sin I$ and ψ would thus reduce to the argument of latitude, i.e., to the angle between the line of nodes and the radius vector to the satellite.

Then

$$\pm G^{-\frac{1}{2}}(\eta)d\eta=(\alpha_2^2-\alpha_3^2)^{-\frac{1}{2}}\eta_0(1-q^2\sin^2\psi)^{-\frac{1}{2}}d\psi, \tag{6.5}$$

where

$$q \equiv \eta_0/\eta_2 = O(k^4) \ll 1, \tag{6.6}$$

η_2 being given by (4.16).

6.1. The Integrals N_1 and N_2

Insertion of (6.4) and (6.5) into (6.1) gives

$$N_1 = (\alpha_2^2 - \alpha_3^2)^{-\frac{1}{2}} \eta_0^3 \int_0^\psi (1 - q^2 \sin^2 \psi)^{-\frac{1}{2}} \sin^2 \psi \, d\psi. \tag{6.7}$$

Because of the identity

$$(1 - q^2 \sin^2 \psi)^{-\frac{1}{2}} q^2 \sin^2 \psi \equiv (1 - q^2 \sin^2 \psi)^{-\frac{1}{2}} - (1 - q^2 \sin^2 \psi)^{\frac{1}{2}}, \tag{6.8}$$

$$N_1 = (\alpha_2^2 - \alpha_3^2)^{-\frac{1}{2}} \eta_0^3 q^{-2} \left[\int_0^\psi (1 - q^2 \sin^2 \psi)^{-\frac{1}{2}} d\psi - \int_0^\psi (1 - q^2 \sin^2 \psi)^{\frac{1}{2}} d\psi \right] \tag{6.9}$$

$$= (\alpha_2^2 - \alpha_3^2)^{-\frac{1}{2}} \eta_0^3 q^{-2} \, [F(\psi, q) - E(\psi, q)], \tag{6.10}$$

where $F(\psi,q)$ and $E(\psi,q)$ are respectively the elliptic integrals of the first and second kinds, with modulus q.

Insertion of (6.4) and (6.5) into (6.2) gives

$$N_2 = (\alpha_2^2 - \alpha_3^2)^{-\frac{1}{2}} \eta_0 \int_0^\psi (1 - q^2 \sin^2 \psi)^{-\frac{1}{2}} d\psi \tag{6.11}$$

$$= (\alpha_2^2 - \alpha_3^2)^{-\frac{1}{2}} \eta_0 F(\psi, q). \tag{6.12}$$

Thus N_1 and N_2 can both be expressed exactly in terms of elliptic integrals of the first and second kinds. Our ultimate purpose, however, is to express each uniformising variable as the sum of an exact secular term and periodic terms correct through $O(k^2)$. For this purpose it is convenient to express each elliptic integral as a linear combination of ψ and a Fourier series $\sum_1^\infty B_n \sin 2n\psi$.

To do so, consider

$$F(\psi, q) \equiv \int_0^\psi (1 - q^2 \sin^2 x)^{-\frac{1}{2}} dx. \tag{6.13}$$

Some simple transformations show that

$$F(\psi + \pi, q) = F(\psi, q) + 2K(q), \tag{6.14}$$

where

$$K(q) \equiv \int_0^{\pi/2} (1 - q^2 \sin^2 x)^{-\frac{1}{2}} dx \tag{6.15}$$

is the complete elliptic integral of the first kind. It follows that the function $F(\psi,q) - (2/\pi)K(q)\psi$ is periodic in ψ with period π.

Furthermore it is an odd function of ψ, so that it can be expanded in a Fourier series containing only the sines of even multiples of ψ. Thus,

$$F(\psi, q) = (2/\pi) K(q) \psi + \sum_{m=1}^{\infty} F_{qm} \sin 2m\psi. \tag{6.16}$$

To calculate the Fourier coefficients F_{qm}, differentiate (6.16) with respect to ψ and use (6.13). Then

$$(1 - q^2 \sin^2 \psi)^{-\frac{1}{2}} = (2/\pi) K(q) + 2 \sum_1^\infty m F_{qm} \cos 2m\psi. \tag{6.17}$$

The Fourier coefficients F_{qm} are then given by

$$F_{qm}=(2/\pi m)\int_0^{\pi/2}(1-q^2\sin^2 x)^{-\frac{1}{2}}\cos 2mx\,dx. \tag{6.18}$$

Since $\cos 2mx$ is a polynomial in $\cos^2 x$, each F_{qm} can ultimately be expressed as a linear combination of $K(q)$ and the complete elliptic integral

$$E(q)\equiv\int_0^{\pi/2}(1-q^2\sin^2 x)^{\frac{1}{2}}dx. \tag{6.19}$$

Such a procedure, however, would not readily reveal the order of each coefficient in q^2, which itself is of order k. (Actually we shall show that F_{qm} is of order k^m, so that we shall need only F_{q1} and F_{q2}.) Instead we expand the radical, obtaining

$$(1-q^2\sin^2 x)^{-\frac{1}{2}}=1+\sum_{n=1}^{\infty}\frac{(2n)!q^{2n}\sin^{2n}x}{2^{2n}(n!)^2}. \tag{6.20}$$

Then

$$F_{qm}=\frac{2}{\pi m}\sum_{n=1}^{\infty}\frac{(2n)!q^{2n}}{2^n(n!)^2}\int_0^{\pi/2}\sin^{2n}x\cos 2mx\,dx. \tag{6.21}$$

Also

$$\sin^{2n}x=\frac{(2n)!}{(n!)^2 2^{2n}}+(-1)^n 2^{1-2n}\sum_{j=0}^{n-1}(-1)^j\frac{(2n)!}{(2n-j)!j!}\cos(2n-2j)x,\qquad(n\geqq 1) \tag{6.22}$$

and

$$\cos[(2n-2j)x]\cos 2mx=\tfrac{1}{2}\cos[(2n+2m-2j)x]+\tfrac{1}{2}\cos[(2n-2m-2j)x]. \tag{6.23}$$

The integral of this product from 0 to $\pi/2$ fails to vanish only if $j=n+m$ or if $j=n-m$. The value $j=n+m$ is absent from (6.22), so that only the term $j=n-m$ in (6.22) contributes to the integral in (6.21). Thus

$$\int_0^{\pi/2}\sin^{2n}x\cos 2mx\,dx=(-1)^m 2^{-2n}\frac{(2n)!}{(n+m)!(n-m)!}\frac{\pi}{2},\qquad(n\geqq m). \tag{6.24}$$

On inserting (6.24) into (6.21), we find

$$F_{qm}=(-1)^m m^{-1}\sum_{n=m}^{\infty}\frac{(2n)!^2 q^{2n}}{(n+m)!(n-m)!2^{4n}(n!)^2}. \tag{6.25}$$

Inspection of (6.25) now shows that $F_{qm}=O(q^{2m})=O(k^m)$. The first two values are, to order q^4 or order k^2:

$$F_{q1}=-\frac{q^2}{8}\left(1+\frac{3}{4}q^2\right)+\ldots, \tag{6.26}$$

$$F_{q2}=\frac{3q^4}{256}+\ldots. \tag{6.27}$$

For our purposes, therefore,

$$F(\psi,q)=(2/\pi)K(q)\psi-\frac{q^2}{8}\left(1+\frac{3}{4}q^2\right)\sin 2\psi+\frac{3q^4}{256}\sin 4\psi+\ldots. \tag{6.28}$$

Similar considerations about periodicity and oddness show that

$$E(\psi,q)\equiv\int_0^{\psi}(1-q^2\sin^2 x)^{\frac{1}{2}}dx=(2/\pi)E(q)\psi+\sum_1^{\infty}E_{qm}\sin 2m\psi. \tag{6.29}$$

Then, as before,

$$(1-q^2\sin^2\psi)^{\frac{1}{2}}=(2/\pi)E(q)+2\sum_1^{\infty}mE_{qm}\cos 2m\psi \tag{6.30}$$

and

$$E_{qm}=(2/\pi m)\int_0^{\pi/2}(1-q^2\sin^2 x)^{\frac{1}{2}}\cos 2mx dx. \tag{6.31}$$

Also

$$(1-q^2\sin^2 x)^{\frac{1}{2}}=1-\sum_{n=1}^{\infty}\frac{(2n-2)!\,q^{2n}\sin^{2n}}{2^{2n-1}n!(n-1)!}x. \tag{6.32}$$

Inserting (6.32) into (6.31), we find

$$E_{qm}=-\frac{1}{\pi m}\sum_{n=1}^{\infty}\frac{(2n-2)!\,q^{2n}}{2^{2n-2}n!(n-1)!}\int_0^{\pi/2}\sin^{2nx}\cos 2mx dx. \tag{6.33}$$

On inserting (6.24) into this, we obtain

$$E_{qm}=\frac{(-1)^{m+1}}{m}\sum_{n=m}^{\infty}\frac{(2n)!\,(2n-2)!\,q^{2n}}{2^{4n-1}n!\,(n-1)!\,(n+m)!\,(n-m)!}. \tag{6.34}$$

Thus E_{qm} is of order q^{2m} or of order k^m and

$$E_{q1}=\frac{q^2}{8}+\frac{q^4}{32}+\ldots, \tag{6.35}$$

$$E_{q2}=-\frac{q^4}{256}+\ldots. \tag{6.36}$$

Then, for our purposes

$$E(\psi,q)=(2/\pi)\,E(q)\,\psi+\left(\frac{q^2}{8}+\frac{q^4}{32}\right)\sin 2\psi-\frac{q^4}{256}\sin 4\psi+\ldots. \tag{6.37}$$

Finally, inserting (6.28) and (6.37) into (6.10) and (6.12), we have for the η-integrals N_1 and N_2:

$$N_1=(\alpha_2^2-\alpha_3^2)^{-\frac{1}{2}}\eta_0^3\left[B_1\psi-\frac{1}{4}\left(1+\frac{1}{2}q^2\right)\sin 2\psi+\frac{q^2}{64}\sin 4\psi+\ldots\right], \tag{6.38}$$

$$N_2=(\alpha_2^2-\alpha_3^2)^{-\frac{1}{2}}\eta_0\left[B_2\psi-\frac{q^2}{32}(4+3q^2)\sin 2\psi+\frac{3q^4}{256}\sin 4\psi+\ldots\right], \tag{6.39}$$

with

$$B_1\equiv\frac{2q^{-2}}{\pi}[K(q)-E(q)]=\frac{1}{2}+\frac{3}{8}q^2+\frac{15}{128}q^4+\ldots, \tag{6.40}$$

$$B_2\equiv\frac{2}{\pi}K(q)=1+\frac{1}{4}q^2+\frac{9}{64}q^4+\ldots. \tag{6.41}$$

Here the terms in ψ are exact. In N_2 the sines are correct to $O(k^2)$, while in N_1 they are correct only to $O(k)$; this is as much accuracy as we need for N_1, however, since it is multiplied by $c^2=kp^2$ in the first kinetic equation (2.5).

6.2. The Integral N_3

From (6.3) and (3.34) we have

$$(\alpha_2^2-\alpha_3^2)^{\frac{1}{2}}N_3=\pm\int_0^{\eta}(1-\eta^2)^{-1}(1-\eta^2/\eta_0^2)^{-\frac{1}{2}}(1-\eta^2/\eta_2^2)^{-\frac{1}{2}}d\eta. \tag{6.42}$$

Then since

$$(1-\eta^2/\eta_2^2)^{-\frac{1}{2}}=\sum_{m=0}^{\infty}\frac{(2m)!}{2^{2m}(m!)^2}(\eta/\eta_2)^{2m}, \tag{6.43}$$

we obtain

$$(\alpha_2^2-\alpha_3^2)^{\frac{1}{2}}N_3=\sum_{m=0}^{\infty}\frac{(2m)!\,\eta_2^{-2m}}{2^{2m}(m!)^2}L_m, \tag{6.44}$$

where

$$L_m\equiv\pm\int_0^{\eta}(1-\eta^2)^{-1}(1-\eta^2/\eta_0^2)^{-\frac{1}{2}}\eta^{2m}d\eta. \tag{6.45}$$

With

$$L_0 \equiv \pm \int_0^{\eta} (1-\eta^2)^{-1}(1-\eta^2/\eta_0^2)^{-\frac{1}{2}} d\eta \tag{6.46}$$

and

$$(1-\eta^2)^{-1}\eta^{2m} \equiv (1-\eta^2)^{-1} - \sum_{n=0}^{m-1} \eta^{2n}, \qquad (m \geqq 1), \tag{6.47}$$

we then find

$$L_m = L_0 - \sum_{n=0}^{m-1} L_{1n}, \qquad (m \geqq 1), \tag{6.48}$$

where

$$L_{1n} = \pm \int_0^{\eta} \eta^{2n}(1-\eta^2/\eta_0^2)^{-\frac{1}{2}} d\eta. \tag{6.49}$$

To evaluate L_0, rewrite (6.46) as

$$L_0 = \pm \int_0^{\eta} \eta^{-3}(\eta^{-2}-1)^{-1}(\eta^{-2}-\eta_0^{-2})^{-\frac{1}{2}} d\eta \tag{6.50}$$

and introduce the new variable χ, defined by the equation

$$\tan \chi = (1-\eta_0^2)^{\frac{1}{2}} \tan \psi = |\cos I| \tan \psi \tag{6.51}$$

and the requirement that χ and ψ shall keep in step. I.e., whenever ψ equals a multiple of $\pi/2$, χ shall be equal to ψ. Then

$$\csc^2 \psi = 1 + \cos^2 I \cot^2 \chi \tag{6.52}$$

and

$$\eta^{-2} = \eta_0^{-2} \csc^2 \psi = 1 + \cot^2 I \csc^2 \chi, \tag{6.53}$$

so that

$$\eta^{-3} d\eta = \cot^2 I \csc^2 \chi \cot \chi \, d\chi \tag{6.54}$$

and $\cot \chi d\chi \gtrless 0$ according as $d\eta \gtrless 0$ in (6.50). With $\eta = \eta_0 \sin \psi$ its integrand becomes

$$\pm |\tan I| \cot \chi \, | \tan \chi \, | \, d\chi = |\tan I| d\chi, \tag{6.55}$$

so that

$$L_0 = |\tan I| \chi = \eta_0 (1-\eta_0^2)^{-\frac{1}{2}} \chi. \tag{6.56}$$

To fit the angle χ into one's knowledge of the corresponding elliptic motion, note that as e approaches zero, η approaches $\sin \theta$. Then, by (6.53),

$$\chi \to \sin^{-1}\left(\frac{\tan \theta}{|\tan I|}\right) = \phi - \Omega, \tag{6.57}$$

where Ω is the right ascension of the node.

To evaluate the integrals L_{1n}, put $\eta = \eta_0 \sin \psi$ in (6.49). Then

$$L_{1n} = \eta_0^{2n+1} \int_0^{\psi} \sin^{2n} x \, dx, \tag{6.58}$$

so that

$$L_{10} = \eta_0 \psi. \tag{6.59}$$

To handle the cases $n \geqq 1$, we rewrite (6.22) as

$$\sin^{2n} x = \frac{(2n)!}{2^{2n}(n!)^2} + 2^{1-2n} \sum_{j=1}^{n} \frac{(-1)^j (2n)!}{(n+j)!(n-j)!} \cos 2jx, \tag{6.60}$$

thus finding

$$L_{1n} = \frac{\eta_0^{2n+1}(2n)!\psi}{2^{2n}(n!)^2} + \eta_0^{2n+1} 2^{-2n} \sum_{j=1}^{n} \frac{(-1)^j (2n)! \sin 2j\psi}{(n+j)!(n-j)!j}, \qquad (n \geqq 1). \tag{6.61}$$

Insertion of (6.48), (6.59), and (6.61) into (6.44) then yields

$$(\alpha_2^2-\alpha_3^2)^{\frac{1}{2}}N_3=L_0\sum_{m=0}^{\infty}\frac{(2m)!\eta_2^{-2m}}{2^{2m}(m!)^2}-\eta_0\psi\sum_{m=1}^{\infty}\frac{(2m)!\eta_2^{-2m}}{2^{2m}(m!)^2}-\psi\sum_{m=2}^{\infty}\frac{(2m)!\eta_2^{-2m}}{2^{2m}(m!)^2}\sum_{n=1}^{m-1}\frac{\eta_0^{2n+1}(2n)!}{2^{2n}(n!)^2}$$
$$-\sum_{m=2}^{\infty}\frac{(2m)!\eta_2^{-2m}}{2^{2m}(m!)^2}\sum_{n=1}^{m-1}\eta_0^{2n+1}2^{-2n}\sum_{j=1}^{n}\frac{(-1)^j(2n)!\sin 2j\psi}{(n+j)!(n-j)!j}. \tag{6.62}$$

If in (6.62) we now use (6.56) and the relation

$$(1-\eta_2^{-2})^{-\frac{1}{2}}=\sum_{m=0}^{\infty}\frac{(2m)!\eta_2^{-2m}}{2^{2m}(m!)^2}, \tag{6.63}$$

we find

$$(\alpha_2^2-\alpha_3^2)^{\frac{1}{2}}N_3=\eta_0[(1-\eta_0^2)^{-\frac{1}{2}}(1-\eta_2^{-2})^{-\frac{1}{2}}\chi+B_3\psi+\sum_{s=1}^{\infty}B_{3s}\sin 2s\psi], \tag{6.64}$$

where

$$B_3=1-(1-\eta_2^{-2})^{-\frac{1}{2}}-\sum_{m=2}^{\infty}\gamma_m\eta_2^{-2m}, \tag{6.65}$$

$$\gamma_m\equiv\frac{(2m)!}{2^{2m}(m!)^2}\sum_{n=1}^{m-1}\frac{(2n)!\eta_0^{2n}}{2^{2n}(n!)^2}, \tag{6.66}$$

and

$$\sum_{s'=1}^{\infty}B_{3s'}\sin 2s'\psi=-\sum_{m=2}^{\infty}\frac{(2m)!\eta_2^{-2m}}{2^{2m}(m!)^2}\sum_{n=1}^{m-1}\eta_0^{2n}2^{-2n}\sum_{j=1}^{n}\frac{(-1)^j(2n)!\sin 2j\psi}{(n+j)!(n-j)!j}. \tag{6.67}$$

The easiest way to isolate the coefficients B_{3s} is to use the orthogonality of the functions $\sin 2s\psi$. Then

$$B_{3s}=-\sum_{m=s+1}^{\infty}\frac{(2m)!\eta_2^{-2m}}{2^{2m}(m!)^2}\sum_{n=s}^{m-1}\frac{\eta_0^{2n}2^{-2n}(-1)^s(2n)!}{(n+s)!(n-s)!s} \tag{6.68}$$

$$=O(\eta_2^{-2s-2}), \tag{6.69}$$

so that $B_{31}=O(\eta_2^{-4})=O(k^2)$, $B_{32}=O(\eta_2^{-6})=O(k^3)$, etc. Thus, to obtain the sine terms of N_3 to $O(k^2)$, we need only

$$B_{31}=\frac{3}{32}\eta_0^2\eta_2^{-4}+\ldots. \tag{6.70}$$

To test the convergence of the series $\sum_2^{\infty}\gamma_m\eta_2^{-2m}$, note first that

$$\frac{(2n)!}{2^{2n}(n!)^2}=\frac{1\cdot 3\cdot 5\ldots(2n-1)}{2\cdot 4\cdot 6\ldots 2n}\leqq\frac{1}{2}, \qquad (n\geqq 1), \tag{6.71}$$

and hence that

$$\sum_{n=1}^{m-1}\frac{(2n)!\eta_0^{2n+1}}{2^{2n}(n!)^2}\leqq\frac{m-1}{2} \tag{6.72}$$

for any orbit, polar or nonpolar. Then, by (6.66), (6.71), and (6.72)

$$\gamma_m\leqq\frac{3}{8}\left(\frac{m-1}{2}\right)\leqq\frac{3}{16}(m-1), \qquad (m\geqq 2). \tag{6.73}$$

Thus $\gamma_m<m$ and

$$\sum_{m=2}^{\infty}\gamma_m\eta_2^{-2m}<\sum_{m=2}^{\infty}m\eta_2^{-2m}. \tag{6.74}$$

But

$$\sum_2^{\infty}m\eta_2^{-2m}=\frac{\eta_2^{-2}}{(1-\eta_2^{-2})^2}-\eta_2^{-2}=\frac{\eta_2^{-4}(2-\eta_2^{-2})}{(1-\eta_2^{-2})^2}, \tag{6.75}$$

where $\eta_2^{-2}=O(k)$. Thus the series $\sum_2^{\infty}\gamma_m\eta_2^{-2m}$ converges rapidly.

To test the convergence of the Fourier series $\sum_{s=1}^{\infty} B_{3s} \sin 2s\psi$, note that

$$\left|\sum_{s=1}^{\infty} B_{3s} \sin 2s\psi\right| \leqq \sum_{s=1}^{\infty} |B_{3s}| \leqq \sum_{m=s+1}^{\infty} \frac{(2m)!\eta_2^{-2m}}{2^{2m}(m!)^2} \sum_{n=s}^{m-1} \frac{(2n)!}{2^{2n}(n+s)!(n-s)!}. \tag{6.76}$$

But

$$\frac{(2n)!}{2^{2n}(n+s)!(n-s)!} = \frac{(2n)!n!n!}{2^{2n}(n!)^2(n+s)!(n-s)!} = \frac{(2n)!}{2^{2n}(n!)^2} \frac{(n)(n-1)(n-2)\ldots(n-s+1)}{(n+s)(n+s-1)(n+s-2)\ldots(n+1)} \leqq \frac{(2n)!}{2^{2n}(n!)^2} < 1,$$

by (6.71). Then

$$\sum_{n=s}^{m-1} \frac{(2n)!}{2^{2n}(n+s)!(n-s)!} < m$$

and

$$\left|\sum_{s=1}^{\infty} B_{3s} \sin 2s\psi\right| < \sum_{m=s+1}^{\infty} \frac{m(2m)!\eta_2^{-2m}}{2^{2m}(m!)^2} < \sum_{m=2}^{\infty} m\eta_2^{-2m}$$

by (6.71) again. Then, by (6.75)

$$\left|\sum_{s=1}^{\infty} B_{3s} \sin 2s\psi\right| < \frac{\eta_2^{-4}(2-\eta_2^{-2})}{(1-\eta_2^{-2})^2}. \tag{6.77}$$

Since $\eta_2^{-2} = O(k)$, the Fourier series converges rapidly.

Thus all terms in (6.64) remain finite for all $\eta_0^2 \leqq 1$, except the term involving χ, which apparently may become infinite for polar orbits. Note, however, that N_3 occurs only in eq (2.7) for the right ascension, when it has a factor α_3, which vanishes for a polar orbit. To see what happens in this case we must investigate the limit of $\alpha_3 N_3$ as $\eta_0^2 \to 1$. For a polar orbit we have

$$\alpha_3 N_3 = \alpha_3(\alpha_2^2 - \tfrac{2}{3}\alpha)^{-\frac{1}{2}}\eta_0(1-\eta_0^2)^{-\frac{1}{2}}(1-\eta_2^{-2})^{-\frac{1}{2}}\chi + O(\alpha_3). \tag{6.78}$$

But $|\alpha_3|(\alpha_2^2 - \alpha_3^2)^{-\frac{1}{2}} = |\alpha_3|\alpha_2^{-1}$ to $O(\alpha_3)$ and by (4.8) and (4.2)

$$|\alpha_3|\alpha_2^{-1} = (1-\eta_0^2)^{\frac{1}{2}}(1-\eta_2^{-2}\sin^2 i_0)^{\frac{1}{2}} = (1-\eta_0^2)^{\frac{1}{2}}(1-\eta_2^{-2})^{\frac{1}{2}} \text{ to } O(\alpha_3). \tag{6.79}$$

Thus to $O(\alpha_3)$

$$\alpha_3 N_3 = (\operatorname{sgn} \alpha_3)\chi \qquad \text{(polar orbit)}. \tag{6.80}$$

If we now use (6.51) to plot χ versus ψ for various values of η_0^2, we obtain figure 1.

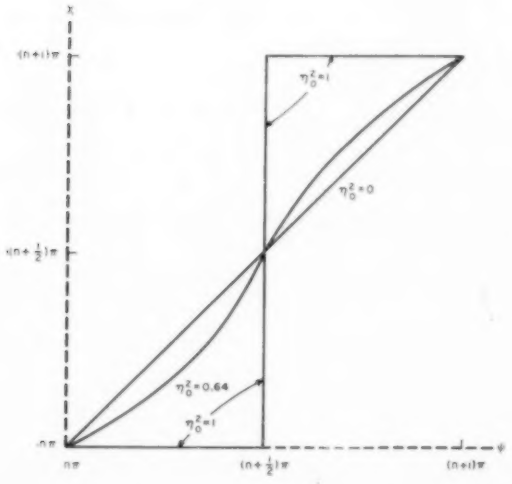

FIGURE 1. *Plot of* χ *versus* ψ *for various values of* η_0^2.

The general form of these curves can be checked, as follows. From (6.51)

$$\tan \chi = \epsilon \tan \psi, \qquad \epsilon \equiv (1-\eta_0^2)^{\frac{1}{2}}. \tag{6.81}$$

To find what happens to a polar orbit we must let ϵ approach zero.
If we put

$$\psi = (n+\tfrac{1}{2})\pi + \Delta\psi, \qquad \chi = (n+\tfrac{1}{2})\pi + \Delta\chi \qquad (n=0,1,2.3, \ldots),$$

then

$$\tan \Delta\chi = \epsilon^{-1} \tan \Delta\psi.$$

If we keep $\Delta\psi$ fixed and let $\epsilon \to 0$, then if $0 < \Delta\psi < \pi/2$, it follows that $\Delta\chi \to \pi/2$. If $-\frac{\pi}{2} < \Delta\psi < 0$, then $\Delta\chi \to -\pi/2$. In a polar orbit, whenever the satellite passes over a pole, $\psi = (n+\frac{1}{2})\pi$, with $\dot{\psi} > 0$, so that χ thus jumps by $+\pi$. By (6.80) and (2.7) the right ascension ϕ then jumps by $+\pi$ in a direct orbit or by $-\pi$ in a retrograde orbit. These are expected results, which had to be obtained as a partial check of eq (6.64) for N_3.

7. Mean Motions

The purposes of this section are to obtain expressions for the mean motions correct through the first order, to find if we are on the right track, and to obtain exact expressions for the mean frequencies for later use in section 8 in checking the secular terms in the final solution.

If p_ρ, p_η, and p_ϕ are the generalized momenta conjugate to ρ, η, and ϕ, then the action variables

$$J_1 \equiv \oint p_\rho d\rho = 2\int_{\rho_1}^{\rho_2} p_\rho d\rho, \tag{7.1}$$

$$J_2 \equiv \oint p_\eta d\eta = 4\int_0^{\eta_0} p_\eta d\eta, \tag{7.2}$$

$$J_3 \equiv \oint p_\phi d\phi = \int_0^{2\pi} p_\phi d\phi = 2\pi\alpha_3, \tag{7.3}$$

are functions of the Jacobi constants α_1, α_2, α_3. (Since these J's occur only in this section, there is no danger of confusion with the coefficients of the zonal harmonics in the expansion of the potential.) The α's are then functions of these J's and the mean frequencies [8] are given by

$$\text{mean } \rho\text{-frequency} = \nu_1 \equiv \partial\alpha_1/\partial J_1, \tag{7.4}$$

$$\text{mean } \eta\text{-frequency} = \nu_2 \equiv \partial\alpha_1/\partial J_2, \tag{7.5}$$

$$\text{mean } \phi\text{-frequency} = \nu_3 \equiv \partial\alpha_1/\partial J_3. \tag{7.6}$$

Note that ν_2 and ν_3 are identical with the usual nodal and sidereal frequencies, but that ν_1 is somewhat different from the usual anomalistic frequency.

To compute these frequencies, one may use the system of equations

$$\sum_{m=1}^{3} \frac{\partial\alpha_l}{\partial J_m}\frac{\partial J_m}{\partial \alpha_n} = \frac{\partial \alpha_l}{\partial \alpha_n} = \delta_{ln}, \qquad (n=1,2,3). \tag{7.7}$$

With the use of (7.4) through (7.6) and the abbreviation

$$J_{mn} \equiv \partial J_m/\partial\alpha_n, \tag{7.8}$$

these equations become

$$\nu_1 J_{11}+\nu_2 J_{21} \qquad =1, \tag{7.9}$$

$$\nu_1 J_{12}+\nu_2 J_{22} \qquad =0, \tag{7.10}$$

$$\nu_1 J_{13}+\nu_2 J_{23}+2\pi\nu_3=0, \tag{7.11}$$

with the solution

$$\nu_1=J_{22}/\Delta \tag{7.12}$$

$$\nu_2=-J_{12}/\Delta \tag{7.13}$$

$$2\pi\nu_3=-\nu_1 J_{13}-\nu_2 J_{23} \tag{7.14}$$

$$\Delta\equiv J_{11}J_{22}-J_{12}J_{21} \tag{7.15}$$

With the use of eqs (13), (53), and (55) of [1] and sections (5) and (6) of the present paper we find

$$J_{11}=2R_1(\rho_2)=2\pi(-2\alpha_1)^{-\frac{1}{2}}(a+b_1+A_1), \tag{7.16}$$

$$J_{12}=-2\alpha_2 R_2(\rho_2)=-2\pi\alpha_2(-2\alpha_1)^{-\frac{1}{2}}A_2, \tag{7.17}$$

$$J_{13}=2c^2\alpha_3 R_3(\rho_2)=2\pi c^2\alpha_3(-2\alpha_1)^{-\frac{1}{2}}A_3, \tag{7.18}$$

$$J_{21}=4c^2 N_1(\eta_0)=2\pi c^2(\alpha_2^2-\alpha_3^2)^{-\frac{1}{2}}\eta_0^3 B_1 \tag{7.19}$$

$$J_{22}=4\alpha_2 N_2(\eta_0)=2\pi\alpha_2(\alpha_2^2-\alpha_3^2)^{-\frac{1}{2}}\eta_0 B_2, \tag{7.20}$$

$$J_{23}=-4\alpha_3 N_3(\eta_0)=-2\pi\alpha_3(\alpha_2^2-\alpha_3^2)^{-\frac{1}{2}}\eta_0[B_3+(1-\eta_0^2)^{-\frac{1}{2}}(1-\eta_2^{-2})^{-\frac{1}{2}}]. \tag{7.21}$$

Then

$$2\pi\nu_1=(-2\alpha_1)^{\frac{1}{2}}[a+b_1+A_1+c^2\eta_0^2 A_2 B_1 B_2^{-1}]^{-1} \tag{7.22}$$

$$2\pi\nu_2=(\alpha_2^2-\alpha_3^2)^{\frac{1}{2}}\eta_0^{-1}A_2 B_2^{-1}[a+b_1+A_1+c^2\eta_0^2 A_2 B_1 B_2^{-1}]^{-1}. \tag{7.23}$$

The above results are all exact, for the potential (2.3), and we shall refer to them again in section 8.

For comparison with other theories it is desirable to express these mean frequencies as power series in the oblateness parameter. For this purpose it is convenient to use the orbital elements a_0, e_0, and i_0, and since we shall carry the series only through the first power we may replace $k_0\equiv c^2/p_0^2$ by $k\equiv c^2/p^2$.

With use of the relations in sections 3 and 4, the above equations then become

$$2\pi\nu_1=n_0+O(k^2) \tag{7.24}$$

$$2\pi\nu_2=n_0[1+\tfrac{3}{4}k(5\cos^2 i_0-1)]+O(k^2), \tag{7.25}$$

where n_0 is given by

$$\mu=n_0^2 a_0^3. \tag{7.26}$$

Similarly

$$J_{13}=\pi k(2+e_0^2)\cos i_0+O(k^2), \tag{7.27}$$

$$J_{23}=-2\pi\ \mathrm{sgn}\ \alpha_3+\pi k(1-e_0^2)\cos i_0+O(k^2), \tag{7.28}$$

which together with (7.14), (7.24), and (7.25) lead to

$$2\pi\nu_3/n_0=(\mathrm{sgn}\ \alpha_3)[1+\tfrac{3}{4}k(5\cos^2 i_0-1)]-\tfrac{3}{2}\,k\cos i_0+O(k^2). \tag{7.29}$$

Here sgn $\alpha_3=\pm 1$ accordingly as the orbit is direct or retrograde, respectively.

To avoid any use of the concept of an osculating ellipse, we may define the mean motions as follows. We say that the ascending node exists only when the satellite is over the equator, travelling north. Let Ω be its right ascension at such a time. We then define the mean

motion $\bar{\dot{\Omega}}$ of the node relative to OX by

$$\bar{\dot{\Omega}} \equiv \lim_{t_i \to \infty} \frac{\Omega_i - \Omega_0}{t_i - t_0}, \tag{7.30}$$

where Ω_0 and t_0 are the values of Ω and t at some (ascending) node and Ω_i and t_i are their values i nodes later. Since the present system is of the conditionally periodic Staeckel type, it follows, after some fairly close reasoning that I shall here omit, that

$$\bar{\dot{\Omega}} = 2\pi(|\nu_3| - \nu_2) \operatorname{sgn} \alpha_3. \tag{7.31}$$

We also say that ρ-perigee exists when, and only when, $\rho = \rho_1$ and we let Φ be its right ascension at such a time. Then the mean motion of ρ-perigee relative to OX must be equal to the mean motion $\bar{\dot{\Phi}}$ of its equatorial projection relative to OX and

$$\bar{\dot{\Phi}} \equiv \lim_{t_i \to \infty} \frac{\Phi_i - \Phi_0}{t_i - t_0} \tag{7.32}$$

Here Φ_0 and t_0 are the values of Φ and t at some ρ-perigee and Φ_i and t_i are their values i ρ-perigees later. Then, again omitting the proof, we have

$$\bar{\dot{\Phi}} = 2\pi(|\nu_3| - \nu_1) \operatorname{sgn} \alpha_3, \tag{7.33}$$

Now let ω be the arc on the celestial sphere from an ascending node (when it exists) to the next ρ-perigee. Then, since the mean relative motion of these points must be equal to the mean relative motion of their projections in the equatorial plane, we find that the mean motion $\bar{\dot{\omega}}$ of ρ-perigee relative to the node is given by

$$\bar{\dot{\omega}} = (\bar{\dot{\Phi}} - \bar{\dot{\Omega}}) \operatorname{sgn} \alpha_3, \tag{7.34}$$

$$= 2\pi(\nu_2 - \nu_1), \tag{7.35}$$

with use of (7.31) and (7.33), for any orbit, direct or retrograde.

From (7.31), (7.25), and (7.29) it follows that

$$\bar{\dot{\Omega}} = -\tfrac{3}{2} k n_0 \cos i_0 + O(k^2) \tag{7.36}$$

and from (7.35), (7.24), and (7.25) that

$$\bar{\dot{\omega}} = \tfrac{3}{4} k n_0 (5 \cos^2 i_0 - 1) + O(k^2) \tag{7.37}$$

Here $n_0/2\pi$ is the frequency in an elliptic orbit with the same total energy. Equation (7.36) agrees with results found by many other authors, as does (7.37) when $\bar{\dot{\omega}}$ is the mean motion of r-perigee relative to the node. By (2.1) and (2.2), however, $r^2 = \rho^2 + kp^2 (1 - \eta^2)$, so that r and ρ differ by a variable quantity of $O(k)$. It is thus a little surprising that the mean motions of ρ-perigee and of r-perigee relative to the node should be equal through $O(k)$. This means that the mean ρ-frequency is equal to the mean r-frequency to this order.

8. Solution of the Kinetic Equations

Before solving the kinetic equations (2.5) through (2.7) it is convenient to have several relations connecting the uniformising variables E and v. From (5.12) we obtain

$$\cos v=\frac{\cos E-e}{1-e\cos E}. \tag{8.1a}$$

The requirements that $dv/dt>0$, $dE/dt>0$ for all t lead to the result that $dv/dE>0$ for all t. Because of this result, (8.1a) leads to

$$\sin v=+\frac{(1-e^2)^{\frac{1}{2}}\sin E}{1-e\cos E}, \tag{8.1b}$$

without ambiguity in sign. For a given value of E, eqs (8.1a,b) determine v modulo 2π. On imposing the further requirement that v shall always equal E whenever the latter is a multiple of π, we find that E determines v completely. Two other relations are often useful, viz,

$$\tan\frac{v}{2}=\left(\frac{1+e}{1-e}\right)^{\frac{1}{2}}\tan\frac{E}{2} \tag{8.1c}$$

and

$$\tan\frac{v-E}{2}=\frac{\gamma\sin E}{1-\gamma\cos E},\qquad \gamma\equiv e^{-1}[1-(1-e^2)^{\frac{1}{2}}]<1. \tag{8.1d}$$

Before beginning the solution of the kinetic equations it is desirable to assemble the results already obtained. By (2.5), (2.6), (5.1), (5.2), (5.30) through (5.33), (5.6), (5.28), (5.35) through (5.40), (6.1), (6.2), (6.6), and (6.38) through (6.41), the equations for ρ and η are

$$l+\beta_1=(-2\alpha_1)^{-\frac{1}{2}}[b_1E+a(E-e\sin E)+A_1v+A_{11}\sin v+A_{12}\sin 2v]$$
$$+e^2(\alpha_2^2-\alpha_3^2)^{-\frac{1}{2}}\eta_0^3\left[B_1\psi-\frac{1}{8}(2+q^2)\sin 2\psi+\frac{q^2}{64}\sin 4\psi\right]+\text{periodic terms of } O(k^3), \tag{8.2}$$

$$\beta_2/\alpha_2=-(-2\alpha_1)^{-\frac{1}{2}}[A_2v+A_{21}\sin v+A_{22}\sin 2v+A_{23}\sin 3v+A_{24}\sin 4v]$$
$$+(\alpha_2^2-\alpha_3^2)^{-\frac{1}{2}}\eta_0\left[B_2\psi-\frac{q^2}{32}(4+3q^2)\sin 2\psi+\frac{3q^4}{256}\sin 4\psi\right]+\text{periodic terms of } O(k^3). \tag{8.3}$$

Here

$$\rho=a(1-e\cos E)=a(1-e^2)(1+e\cos v)^{-1}, \tag{8.4}$$

$$\eta=\eta_0\sin\psi, \tag{8.5}$$

with E and v connected by any of the eqs (8.1).

$$A_1=(1-e^2)^{\frac{1}{2}}p\sum_{n=2}^{\infty}(b_2/p)^nP_n(b_1/b_2)R_{n-2}[(1-e^2)^{\frac{1}{2}}]=O(k), \tag{8.6}$$

$$A_2=(1-e^2)^{\frac{1}{2}}p^{-1}\sum_{n=0}^{\infty}(b_2/p)^nP_n(b_1/b_2)R_n[(1-e^2)^{\frac{1}{2}}]=O(k^0), \tag{8.7}$$

$$A_{11}=\tfrac{3}{4}(1-e^2)^{\frac{1}{2}}p^{-3}e(-2b_1b_2^2p+b_2^4)=O(k^2), \tag{8.8}$$

$$A_{12}=\frac{3}{32}(1-e^2)^{\frac{1}{2}}p^{-3}b_2^4e^2=O(k^2), \tag{8.9}$$

$$B_1=\frac{2}{\pi q^2}[K(q)-E(q)]=O(k^0), \tag{8.10}$$

$$B_2=\frac{2}{\pi}K(q)=O(k^0), \tag{8.11}$$

$$q=\eta_0/\eta_2=O(k^{\frac{1}{2}}). \tag{8.12}$$

$K(q)$ and $E(q)$ are the complete elliptic integrals of the first and second kinds.

$$A_{21}=(1-e^2)^{\frac{1}{2}}ep^{-5}\left[-\frac{1}{2}Ap^3+\left(\frac{3}{4}A^2-B\right)p^2+\frac{9}{4}ABp\left(1+\frac{e^2}{4}\right)+\frac{3}{8}B^2(4+3e^2)\right]=O(k), \quad (8.13)$$

$$A_{22}=(1-e^2)^{\frac{1}{2}}p^{-5}\left[\left(\frac{3}{8}A^2-\frac{1}{2}B\right)\frac{e^2}{4}p^2+\frac{9}{16}ABe^2p+\frac{3}{8}B^2\left(\frac{3}{2}e^2+\frac{1}{4}e^4\right)\right]=O(k), \quad (8.14)$$

$$A_{23}=(1-e^2)^{\frac{1}{2}}p^{-5}e\left[\frac{ABe^2p}{16}+\frac{B^2e^2}{8}\right]=O(k^2), \quad (8.15)$$

$$A_{24}=\frac{3}{256}(1-e^2)^{\frac{1}{2}}p^{-5}B^2e^4=O(k^2). \quad (8.16)$$

In the above equations $c^2=r_e^2J_2$ is regarded as known, as are the orbital elements a, e, η_0, β_1, and β_2. Then $p=a(1-e^2)$ and η_2, A, B, α_1, α_2, and α_3 are given by section 4. Also $b_1=-\frac{1}{2}A$, $b_2^2=B$, $R_m(x)=x^mP_m(1/x)$, with $|x|<1$.

To solve (8.2) and (8.3), place

$$E=E_s+E_p, \qquad v=v_s+v_p, \qquad \psi=\psi_s+\psi_p, \quad (8.17)$$

where the subscript s means "secular" and the subscript p "periodic." Then if ρ goes through N_1 cycles in time T_1 and if η goes through N_2 cycles in time T_2,

$$\bar{\dot{E}}=\bar{\dot{v}}=\dot{E}_s=\dot{v}_s=\lim_{T_1\to\infty}\frac{2\pi N_1}{T_1}=2\pi\nu_1 \quad (8.18)$$

$$\bar{\dot{\psi}}=\dot{\psi}_s=\lim_{T_2\to\infty}\frac{2\pi N_2}{T_2}=2\pi\nu_2. \quad (8.19)$$

Since we have already obtained exact expressions in section 7 for ν_1 and ν_2, it is clear that we can obtain the secular terms exactly. We shall also obtain the periodic terms through $O(k^2)$.

By (8.18) we can write

$$E_s=v_s=M_s, \quad (8.20)$$

where M_s will play the role of the secular part of a mean anomaly.
Then

$$E=M_s+E_p, \qquad v=M_s+v_p, \qquad \psi=\psi_s+\psi_p. \quad (8.21)$$

We may obtain the secular solution of (8.2) and (8.3) independently of section 7, by dropping all the sines in those equations, placing $E=v=M_s$ and $\psi=\psi_s$, and solving the resulting equations for M_s and ψ_s. The resulting equations are

$$(-2\alpha_1)^{-\frac{1}{2}}(a+b_1+A_1)M_s+c^2(\alpha_2^2-\alpha_3^2)^{-\frac{1}{2}}\eta_0^3B_1\psi_s=t+\beta_1, \quad (8.22)$$

$$-(-2\alpha_1)^{-\frac{1}{2}}A_2M_s+(\alpha_2^2-\alpha_3^2)^{-\frac{1}{2}}\eta_0B_2\psi_s=\beta_2/\alpha_2, \quad (8.23)$$

with the following solution.

8.1. Secular Solution

$$M_s=(-2\alpha_1)^{\frac{1}{2}}\frac{B_2(t+\beta_1)-c^2\beta_2\alpha_2^{-1}\eta_0^2B_1}{(a+b_1+A_1)B_2+c^2\eta_0^2A_2B_1}, \quad (8.24)$$

$$\psi_s=(\alpha_2^2-\alpha_3^2)^{\frac{1}{2}}\eta_0^{-1}\frac{A_2(t+\beta_1)+\beta_2\alpha_2^{-1}(a+b_1+A_1)}{(a+b_1+A_1)B_2+c^2\eta_0^2A_2B_1}. \quad (8.25)$$

Comparison of these results with (7.22) and (7.23) shows that $\dot{M}_s=2\pi\nu_1$ and $\dot{\psi}_s=2\pi\nu_2$, as expected. We may now rewrite these expressions more conveniently as

$$M_s=2\pi\nu_1(t+\beta_1-c^2\beta_2\alpha_2^{-1}\eta_0^2B_1B_2^{-1}), \quad (8.26)$$

$$\psi_s = 2\pi\nu_2 [t + \beta_1 + \beta_2 \alpha_2^{-1}(a + b_1 + A_1) A_2^{-1}]. \tag{8.27}$$

As a check, note that to $O(k^0)$, $2\pi\nu_1 = 2\pi\nu_2 = n_0$, so that

$$M_s = n_0(t + \beta_1) + O(k),$$

$$\psi_s = n_0(t + \beta_1) + \frac{n_0 a \beta_2}{\alpha_2 A_2} + O(k).$$

But, to order k^0, $\alpha_2 = (\mu p)^{\frac{1}{2}}$ and $A_2 = (1-e^2)^{\frac{1}{2}} p^{-1}$, so that $n_0 a(\alpha_2 A_2)^{-1} = n_0 a p[\mu p(1-e^2)]^{-\frac{1}{2}} = 1$, since $n_0^2 a^3 = \mu$ to $O(k^0)$.
Thus

$$\psi_s = M_s + \beta_2 + O(k),$$

as is to be expected, with β_2 replacing ω.

As a later aid in reducing the solution for the periodic terms to Kepler's equation, it is here convenient to rewrite (8.22), by transposition of some of its terms:

$$t + \beta_1 - (-2\alpha_1)^{-\frac{1}{2}}(a + b_1) M_s = (-2\alpha_1)^{-\frac{1}{2}} A_1 M_s + c^2(\alpha_2^2 - \alpha_3^2)^{-\frac{1}{2}} \eta_0^3 B_1 \psi_s. \tag{8.28}$$

8.2. Periodic Terms

We shall put, successively,

$$E_p = E_0, \qquad v_p = v_0, \qquad \psi_p = \psi_0; \tag{Step 0}$$

$$E_p = E_0 + E_1, \qquad v_p = v_0 + v_1, \qquad \psi_p = \psi_0 + \psi_1; \tag{Step 1}$$

$$E_p = E_0 + E_1 + E_2, \qquad v_p = v_0 + v_1 + v_2, \qquad \psi_p = \psi_0 + \psi_1 + \psi_2. \tag{Step 2}$$

In step 0, we retain in the equations (8.2) and (8.3) only those periodic terms which are of (Ok^0), viz, sin E. In step 1, we retain in these equations all periodic terms of order k^0 or k, but none of higher order. In step 2, we retain in the equations all periodic terms of order k^0, k, or k^2, but none of higher order. In carrying out each step, however, we shall suppose that each quantity involved is calculated to such an accuracy that the error is of order k^3. Then, effectively, E_0, v_0, and ψ_0 will all contain terms of order k and k^2, as well as terms of order k^0. E_1, v_1, and ψ_1 will contain no terms of order k^0, but will contain terms of order k and k^2. E_2, v_2, and ψ_2 will be of order k^2. Such a procedure will greatly simplify the resulting equations.

8.3. The Periodic Contributions E_0, v_0, and ψ_0

On placing $E = M_s + E_0$, $v = M_s + v_0$, and $\psi = \psi_s + \psi_0$ in (8.2) and (8.3) and retaining only the term sin E of the periodic terms, we find

$$t + \beta_1 = (-2\alpha_1)^{-\frac{1}{2}}[(a + b_1)(M_s + E_0) - ae \sin(M_s + E_0) + A_1 M_s] + c^2(\alpha_2^2 - \alpha_3^2)^{-\frac{1}{2}} \eta_0^3 B_1 \psi_s, \tag{8.29}$$

$$\beta_2/\alpha_2 = -(-2\alpha_1)^{-\frac{1}{2}} A_2(M_s + v_0) + (\alpha_2^2 - \alpha_3^2)^{-\frac{1}{2}} \eta_0 B_2(\psi_s + \psi_0). \tag{8.30}$$

On subtracting (8.28) from (8.29) and dividing the resulting equation by $(a + b_1)(-2\alpha_1)^{-\frac{1}{2}}$, we find

$$M_s + E_0 - e' \sin(M_s + E_0) = M_s, \tag{8.31}$$

where

$$e' \equiv \frac{ae}{a + b_1} < 1, \tag{8.32}$$

since $b_1 > 0$. Equation (8.31) is Kepler's equation for $M_s + E_0$, with an effective eccentricity e'. Let us suppose it to be solved by the most appropriate method, which will depend on the value of e'. We then have $M_s + E_0$ and can then find $v = M_s + v_0$ by use of eqs (8.1).

On subtracting (8.23) from (8.30) we then obtain ψ_0 as a function of v_0:

$$\psi_0=(-2\alpha_1)^{-\frac{1}{2}}(\alpha_2^2-\alpha_3^2)^{\frac{1}{2}}\eta_0^{-1}A_2B_2^{-1}v_0. \tag{8.33}$$

Here the coefficient ψ_0/v_0 is unity to $O(k^0)$, but to follow the procedure outlined above we must not make such an approximation. Instead we must calculate it so accurately that the error is of order k^3.

8.4. The Periodic Contributions E_1, v_1, and ψ_1

Now, knowing M_s, ψ_s, E_0, v_0, and ψ_0, we place $E=M_s+E_0+E_1$, $v=M_s+v_0+v_1$, and $\psi=\psi_s+\psi_0+\psi_1$ into eqs (8.2) and (8.3), discarding only the periodic terms of order k^2. Then

$$t+\beta_1=(-2\alpha_1)^{-\frac{1}{2}}[(a+b_1)(M_s+E_0+E_1)-ae\sin(M_s+E_0+E_1)+A_1(M_s+v_0)]$$
$$+c^2(\alpha_2^2-\alpha_3^2)^{-\frac{1}{2}}\eta_0^3[B_1(\psi_s+\psi_0)-\tfrac{1}{4}\sin(2\psi_s+2\psi_0)], \tag{8.34}$$

$$\beta_2/\alpha_2=-(-2\alpha_1)^{-\frac{1}{2}}[A_2(M_s+v_0+v_1)+A_{21}\sin(M_s+v_0)+A_{22}\sin(2M_s+2v_0)]$$
$$+(\alpha_2^2-\alpha_3^2)^{-\frac{1}{2}}\eta_0\left[B_2(\psi_s+\psi_0+\psi_1)-\frac{q^2}{8}\sin(2\psi_s+2\psi_0)\right]. \tag{8.35}$$

Subtraction of (8.28) from (8.34) and division of the result by $(a+b_1)(-2\alpha_1)^{-\frac{1}{2}}$ now gives a Kepler equation for $M_s+E_0+E_1$:

$$M_s+E_0+E_1-e'\sin(M_s+E_0+E_1)=M_s+M_1. \tag{8.36}$$

where

$$M_1\equiv(a+b_1)^{-1}\left[-(A_1+c^2\eta_0^2A_2B_1B_2^{-1})v_0+\frac{c^2}{4}(-2\alpha_1)^{\frac{1}{2}}(\alpha_2^2-\alpha_3^2)^{-\frac{1}{2}}\eta_0^3\sin(2\psi_s+2\psi_0)\right]. \tag{8.37}$$

Here we have used (8.33) to combine terms in v_0 and ψ_0. The quantity M_s+M_1 is then a mean anomaly whose secular part is exact and whose periodic part is correct through order k. It has no periodic part of order k^0; this is characteristic of a mean anomaly.

It is not necessary to solve the Kepler equation all over again. If in (8.36) we put

$$\sin(M_s+E_0+E_1)=(1-\tfrac{1}{2}E_1^2)\sin(M_s+E_0)+E_1\cos(M_s+E_0)+O(E_1^3), \tag{8.38}$$

the error is of order k^3. Then (8.36) and (8.38) yield a quadratic equation for E_1, whose solution through terms of $O(k^2)$ is given by

$$E_1=\frac{M_1}{1-e'\cos(M_s+E_0)}-\frac{e'}{2}\frac{M_1^2\sin(M_s+E_0)}{[1-e'\cos(M_s+E_0)]^3}. \tag{8.39}$$

To find v_1 insert $v=M_s+v_0+v_1$ and $E=M_s+E_0+E_1$ into eqs (8.1) and solve for v_1.

On subtracting (8.23) from (8.35) and eliminating terms in v_0 and ψ_0 by use of (8.33), we then find

$$\psi_1=(-2\alpha_1)^{-\frac{1}{2}}(\alpha_2^2-\alpha_3^2)^{\frac{1}{2}}\eta_0^{-1}B_2^{-1}[A_2v_1+A_{21}\sin(M_s+v_0)+A_{22}\sin(2M_s+2v_0)] \tag{8.40}$$
$$+\frac{q^2}{8}B_2^{-1}\sin(2\psi_s+2\psi_0).$$

(Note that the elimination of v_0 and ψ_0 would not have been possible if they had been carried only through order k^0.)

8.5. The Periodic Contributions E_2, v_2, and ψ_2

Finally, knowing M_s, ψ_s, E_0, v_0, ψ_0, E_1, v_1, and ψ_1, we place $E=M_s+E_0+E_1+E_2$, $v=M_s+v_0+v_1+v_2$, and $\psi=\psi_s+\psi_0+\psi_1+\psi_2$ in (8.2) and (8.3), discarding only the periodic terms of order greater than k^2. The equations become

$$t+\beta_1=(-2\alpha_1)^{-\frac{1}{2}}[(a+b_1)(M_s+E_0+E_1+E_2)-ae\sin(M_s+E_0+E_1+E_2)+A_1(M_s+v_0+v_1)$$
$$+A_{11}\sin(M_s+v_0)+A_{12}\sin(2M_s+2v_0)]+c^2(\alpha_2^2-\alpha_3^2)^{-\frac{1}{2}}\eta_0^3\Big[B_1(\psi_s+\psi_0+\psi_1)$$
$$-\frac{1}{8}(2+q^2)\sin(2\psi_s+2\psi_0+2\psi_1)+\frac{q^2}{64}\sin(4\psi_s+4\psi_0)\Big], \tag{8.41}$$

$$\beta_2/\alpha_2=-(-2\alpha_1)^{-\frac{1}{2}}[A_2(M_s+v_0+v_1+v_2)+A_{21}\sin(M_s+v_0+v_1)+A_{22}\sin(2M_s+2v_0+2v_1)$$
$$+A_{23}\sin(3M_s+3v_0)+A_{24}\sin(4M_s+4v_0)]+(\alpha_2^2-\alpha_3^2)^{-\frac{1}{2}}\eta_0\Big[B_2(\psi_s+\psi_0+\psi_1+\psi_2)$$
$$-\frac{q^2}{32}(4+3q^2)\sin(2\psi_s+2\psi_0+2\psi_1)+\frac{3}{256}q^4\sin(4\psi_s+4\psi_0)\Big]. \tag{8.42}$$

On subtracting (8.34) from (8.41) and discarding periodic terms of order k^3 or higher, we find

$$0=(-2\alpha_1)^{-\frac{1}{2}}[(a+b_1)E_2-aeE_2\cos(M_s+E_0+E_1)+A_1v_1+A_{11}\sin(M_s+v_0)$$
$$+A_{12}\sin(2M_s+2v_0)]+c^2(\alpha_2^2-\alpha_3^2)^{-\frac{1}{2}}\eta_0^3\Big[B_1\psi_1-\frac{1}{2}\psi_1\cos(2\psi_s+2\psi_0)$$
$$-\frac{q^2}{8}\sin(2\psi_s+2\psi_0)+\frac{q^2}{64}\sin(4\psi_s+4\psi_0)\Big]. \tag{8.43}$$

Then

$$E_2=\frac{M_2}{1-e'\cos(M_s+E_0+E_1)}, \tag{8.44}$$

where

$$M_2\equiv-(a+b_1)^{-1}\Big[A_1v_1+A_{11}\sin(M_s+v_0)+A_{12}\sin(2M_s+2v_0)$$
$$+c^2(-2\alpha_1)^{\frac{1}{2}}(\alpha_2^2-\alpha_3^2)^{-\frac{1}{2}}\eta_0^3\Big\{B_1\psi_1-\frac{1}{2}\psi_1\cos(2\psi_s+2\psi_0)-\frac{q^2}{8}\sin(2\psi_s+2\psi_0)+\frac{q^2}{64}\sin(4\psi_s+4\psi_0)\Big\}\Big]. \tag{8.45}$$

It is easy to show that, to order k^2

$$M_s+E_0+E_1+E_2-e'\sin(M_s+E_0+E_1+E_2)=M_s+M_1+M_2, \tag{8.46}$$

so that M_2 is the second-order periodic term of a total mean anomaly

$$M=M_s+M_1+M_2+\dots \tag{8.47}$$

corresponding to the effective eccentricity e'.

To find v_2 insert $v=M_s+v_0+v_1+v_2$ and $E=M_s+E_0+E_1+E_2$ into eqs (8.1) and solve for v_2.

To find ψ_2, subtract (8.35) from (8.42) and discard periodic terms of order k^3 or higher. The result is

$$\psi_2=(-2\alpha_1)^{-\frac{1}{2}}(\alpha_2^2-\alpha_3^2)^{\frac{1}{2}}\eta_0^{-1}B_2^{-1}[A_2v_2+A_{21}v_1\cos(M_s+v_0)+2A_{22}v_1\cos(2M_s+2v_0)+A_{23}\sin(3M_s+3v_0)$$
$$+A_{24}\sin(4M_s+4v_0)]+\frac{q^2}{4}B_2^{-1}\Big[\psi_1\cos(2\psi_s+2\psi_0)+3\frac{q^2}{8}\sin(2\psi_2+2\psi_0)-3\frac{q^2}{64}\sin(4\psi_s+4\psi_0)\Big]. \tag{8.48}$$

This completes the solution, with exact secular terms and periodic terms correct through order k^2, for E, v, and ψ and thus for the spheroidal coordinates ρ and η.

8.6. The Right Ascension ϕ

From (2.7), (5.3), and (6.3) we obtain

$$\phi=\beta_3+\alpha_3 N_3-c^2\alpha_3 R_3. \tag{8.49}$$

Then, from the equations in sections 5 and 6

$$\phi=\beta_3+\alpha_3(\alpha_2^2-\alpha_3^2)^{-\frac{1}{2}}\eta_0\Big[(1-\eta_0^2)^{-\frac{1}{2}}(1-\eta_2^{-2})^{-\frac{1}{2}}\chi+B_3\psi$$
$$+\frac{3}{32}\eta_0^2\eta_2^{-4}\sin 2\psi\Big]-c^2\alpha_3(-2\alpha_1)^{-\frac{1}{2}}\Big[A_3 v+\sum_{n=1}^{4}A_{3n}\sin nv\Big], \tag{8.50}$$

where χ is an angle that always equals ψ whenever ψ is a multiple of $\pi/2$ and which also satisfies

$$\tan\chi=(1-\eta_0^2)^{\frac{1}{2}}\tan\psi.$$

The expressions for A_3 and the A_{3n}'s are given in (5.61) to (5.65) and that for B_3 is given in (6.65). With the secular parts of v and ψ exact and their periodic terms correct through order k^2, the right ascension ϕ, as given by (8.50), has a secular part that is exact and a periodic part correct through order k^2. To check the secular part of ϕ, note that one can obtain it from (8.50) by placing $\chi=\psi=\psi_s$ and $v=v_s$ and discarding the sines. If we do so and also use $\dot{v}_s=2\pi\nu_1$ and $\dot{\psi}_s=2\pi\nu_2$, we find

$$\dot{\phi}_s=-2\pi c^2\alpha_3(-2\alpha_1)^{-\frac{1}{2}}A_3\nu_1+2\pi\alpha_3(\alpha_2^2-\alpha_3^2)^{-\frac{1}{2}}\eta_0[B_3+(1-\eta_0^2)^{-\frac{1}{2}}(1-\eta_2^{-2})^{-\frac{1}{2}}]\nu_2=-\nu_1 J_{13}-\nu_2 J_{23} \tag{8.51}$$

on comparison with (7.18) and (7.21). Thus we find

$$\bar{\dot{\phi}}=\dot{\phi}_s=2\pi\nu_3 \tag{8.52}$$

by (8.51) and (7.14), a result known to be correct [8].

A summary of the principal results of the paper follows in section 9.

9. Summary of the Solution

We assume that μ and c are known, where μ is the product of the gravitational constant and the mass of the planet and where $c^2=r_e^2 J_2$, r_e being the equatorial radius and J_2 the coefficient of the second zonal harmonic of the planet's gravitational potential. For the earth $J_2\approx(1.08)10^{-3}$.

If X, Y, Z are the usual rectangular coordinates of an artificial satellite and if r, θ, ϕ are respectively its planetocentric distance, declination, and right ascension, its oblate spheroidal coordinates ρ, η, ϕ are given by

$$X+iY=r\cos\theta\exp i\phi=[(\rho^2+c^2)(1-\eta^2)]^{\frac{1}{2}}\exp i\phi,$$

$$Z=r\sin\theta=\rho\eta,\qquad(-1\leqq\eta\leqq 1).$$

The potential

$$V_a=-\mu\rho\,(\rho^2+c^2\eta^2)^{-1}$$

then fits the even zonal harmonics exactly through the second and, in the case of the earth, approximately through the fourth. Solution for the motion with such a potential thus furnishes a very accurate intermediary orbit. Since this potential leads to separability of the Hamilton-Jacobi equation, the solution is given implicitly by the quadratures of eqs (2.5) through (2.7). The integration constants are the Jacobi α's and β's.

If the initial conditions are known, one can readily evaluate the α's. Then if one can evaluate the integrals in (2.5) through (2.7) one can also evaluate the β's. Evaluating the integrals depends on factoring the quartics $F(\rho)$ and $G(\eta)$. The factoring of $G(\eta)$ is immediate,

since it is quadratic in η^2. To discuss the factoring of $F(\rho)$ we introduce, in place of the α's, the orbital elements $a_0 \equiv -\mu/2\alpha_1$, $\epsilon_0 \equiv [1+2\alpha_1\alpha_2^2/\mu^2]^{\frac{1}{2}}$, and $i_0 \equiv \cos^{-1}(\alpha_3/\alpha_2)$.

With ρ varying in the range $\rho_1 \leqq \rho \leqq \rho_2$, we write

$$F(\rho) \equiv c^2\alpha_3^2 + (\rho^2+c^2)(-\alpha_2^2+2\mu\rho+2\alpha_1\rho^2) = (-2\alpha_1)(\rho-\rho_1)(\rho_2-\rho)(\rho^2+A\rho+B)$$

and find that A, B, a, p, and ϵ occur in the ρ-integrals.
Here

$$a \equiv \frac{1}{2}(\rho_1+\rho_2), \qquad 1-\epsilon^2 = \frac{4\rho_1\rho_2}{(\rho_1+\rho_2)^2}, \qquad p \equiv a(1-\epsilon^2) = \frac{2\rho_1\rho_2}{\rho_1+\rho_2}, \qquad \epsilon \equiv \frac{\rho_2-\rho_1}{\rho_2+\rho_1} = [1-(1-\epsilon^2)]^{\frac{1}{2}}$$

Thus A, B, $\rho_1+\rho_2$, and $\rho_1\rho_2$ are needed. If we put

$$p_0 \equiv a_0(1-\epsilon_0^2), \qquad k_0 \equiv c^2/p_0^2, \qquad x \equiv (1-\epsilon_0^2)^{\frac{1}{2}}, \qquad y \equiv \alpha_3/\alpha_2 = \cos i_0,$$

then through $O(k_0^2)$

$$A = -2k_0p_0y^2[1+k_0(2x^2-3x^2y^2-4+8y^2)+\ldots],$$

$$B = k_0p_0^2(1-y^2)[1+k_0(4y^2-x^2y^2)+\ldots],$$

$$2a \equiv \rho_1+\rho_2 = 2p_0x^{-2}[1-k_0x^2y^2-k_0^2x^2y^2(2x^2-3x^2y^2-4+8y^2)+\ldots],$$

$$ap \equiv \rho_1\rho_2 = p_0^2x^{-2}[1+k_0y^2(x^2-4)-k_0^2y^2(12x^2-x^4-20x^2y^2-16+32y^2+x^4y^2)+\ldots].$$

If a_0, ϵ_0, i_0, β_1, β_2, β_3 are used as orbital elements, we then assume that the quartic $F(\rho)$ is factored numerically, to as great an accuracy as may be desired, with the aid of the above relations as a starting point.

With η varying in the range $-1 \leqq -\eta_0 \leqq \eta \leqq \eta_0 \leqq 1$, we write $G(\eta) \equiv -\alpha_3^2+(1-\eta^2)(\alpha_2^2+2\alpha_1c^2\eta^2)$ $= -2\alpha_1c^2(\eta_0^2-\eta^2)(\eta_2^2-\eta^2)$ and find

$$\begin{pmatrix}\eta_0^{-2}\\ \eta_2^{-2}\end{pmatrix} = \frac{\alpha_2^2-2\alpha_1c^2}{2(\alpha_2^2-\alpha_3^2)}\left[1 \pm \left\{1+\frac{8\alpha_1c^2(\alpha_2^2-\alpha_3^2)}{(\alpha_2^2-2\alpha_1c^2)^2}\right\}^{\frac{1}{2}}\right].$$

Here $\eta_2 \gg 1$. Then all the quantities a, p, ϵ, A, B, η_0, and η_2 are known in terms of the orbital elements a_0, ϵ_0, and i_0.

If we assume that the orbital elements are to be determined by an iterated least-square fitting of the solution to many revolutions in the orbit and not by initial conditions, there is a better set of elements, introduced by Izsak [4]. These, viz, a, ϵ, $I \equiv \sin^{-1}\eta_0$, β_1, β_2, and β_3, although not easily found from the initial conditions, result in immediate factoring of the quartics $F(\rho)$ and $G(\eta)$. We therefore give the solution in terms of these quantities, with the understanding that they are to be determined either by the least-square fitting or from initial conditions by numerical factoring of $F(\rho)$.

Given μ, c, and the elements a, ϵ, I, β_1, β_2, β_3,
compute

$$\eta_0 = \sin I,$$

$$p = a(1-\epsilon^2), \qquad D = (ap-c^2)(ap-c^2\eta_0^2)+4a^2c^2\eta_0^2, \qquad D' = D+4a^2c^2(1-\eta_0^2),$$

$$A = -2ac^2D^{-1}(1-\eta_0^2)(ap-c^2\eta_0^2), \qquad B = c^2\eta_0^2D^{-1}D',$$

$$b_1 = -\frac{1}{2}A, \qquad b_2 = B^{\frac{1}{2}}, \qquad -2\alpha_1 = \mu(a+b_1)^{-1},$$

$$-\frac{\alpha_2^2}{2\alpha_1} = a_0p_0 = -c^2(1-\eta_0^2)+apD^{-1}D', \qquad \alpha_2 = (-2\alpha_1)^{\frac{1}{2}}(a_0p_0)^{\frac{1}{2}} > 0,$$

$$\alpha_3 = \alpha_2\left(1-\frac{c^2\eta_0^2}{a_0p_0}\right)^{\frac{1}{2}}\cos I, \qquad \eta_2^{-2} = \frac{c^2D}{apD'}, \qquad k = c^2/p^2, \qquad q = \eta_0/\eta_2.$$

Restrict considerations to the case $b_1/b_2 \leqq 1$. Then

$$I_c \leqq I \leqq 180^\circ - I_c,$$

where $I_c = 1^\circ 54'$, approximately. Equatorial and almost equatorial orbits are thus ruled out.

Then compute

$$A_1 = (1-e^2)^{\frac{1}{2}} p \sum_{n=2}^{\infty} (b_2/p)^n P_n(b_1/b_2) R_{n-2}[(1-e^2)^{\frac{1}{2}}],$$

$$A_2 = (1-e^2)^{\frac{1}{2}} p^{-1} \sum_{n=0}^{\infty} (b_2/p)^n P_n(b_1/b_2) R_n[(1-e^2)^{\frac{1}{2}}],$$

where $P_n(x)$ is the Legendre polynomial of degree n and where $R_n(x) \equiv x^n P_n(x^{-1})$, always a polynomial of degree $[n/2]$ in x^2.

$$A_3 = (1-e^2)^{\frac{1}{2}} p^{-3} \sum_{m=0}^{\infty} D_m R_{m+2}[(1-e^2)^{\frac{1}{2}}],$$

where

$$D_{2i} = \sum_{n=0}^{i} (-1)^{i-n} (c/p)^{2i-2n} (b_2/p)^{2n} P_{2n}(b_1/b_2),$$

$$D_{2i+1} = \sum_{n=0}^{i} (-1)^{i-n} (c/p)^{2i-2n} (b_2/p)^{2n+1} P_{2n+1}(b_1/b_2),$$

$$B_1 = 2\pi^{-1} q^{-2}[K(q) - E(q)], \qquad B_2 = 2\pi^{-1} K(q),$$

where $K(q)$ and $E(q)$ are the complete elliptic integrals of the first and second kinds, respectively.

$$B_3 = 1 - (1-\eta_2^{-2})^{\frac{1}{2}} - \sum_{m=2}^{\infty} \gamma_m \eta_2^{-2m}, \qquad \text{where} \qquad \gamma_m \equiv \frac{(2m)!}{2^{2m}(m!)^2} \sum_{n=1}^{m-1} \frac{(2n)! \eta_0^{2n}}{2^{2n}(n!)^2}.$$

(The above series all converge rapidly.)

$$A_{11} = \frac{3}{4}(1-e^2)^{\frac{1}{2}} p^{-3} e(-2b_1 b_2^2 p + b_2^4), \qquad A_{12} = \frac{3}{32}\, p^{-3}(1-e^2)^{\frac{1}{2}} b_2^4 e^2,$$

$$A_{21} = (1-e^2)^{\frac{1}{2}} p^{-1} e \left[b_1 p^{-1} + (3b_1^2 - b_2^2) p^{-2} - \frac{9}{2} b_1 b_2^2 \left(1 + \frac{e^2}{4}\right) p^{-3} + \frac{3}{8} b_2^4 (4+3e^2)\, p^{-4} \right],$$

$$A_{22} = (1-e^2)^{\frac{1}{2}} p^{-1} \left[\frac{e^2}{8} (3b_1^2 - b_2^2) p^{-2} - \frac{9}{8}\, e^2 b_1 b_2^2 p^{-3} + \frac{3}{32}\, b_2^4 (6e^2 + e^4)\, p^{-4} \right],$$

$$A_{23} = (1-e^2)^{\frac{1}{2}} p^{-1} \frac{e^3}{8} (-b_1 b_2^2 p^{-3} + b_2^4 p^{-4}),$$

$$A_{24} = \frac{3}{256}\, (1-e^2)^{\frac{1}{2}} p^{-5} b_2^4 e^4,$$

$$A_{31} = (1-e^2)^{\frac{1}{2}} p^{-3} e\, [2 + b_1 p^{-1} (3 + \tfrac{3}{4} e^2) - p^{-2} (\tfrac{1}{2} b_2^2 + c^2)\, (4+3e^2)],$$

$$A_{32} = (1-e^2)^{\frac{1}{2}} p^{-3} \left[\frac{e^2}{4} + \frac{3}{4}\, b_1 p^{-1} e^2 - p^{-2} \left(\frac{e^4}{4} + \frac{3}{2} e^2\right) \left(\frac{1}{2} b_2^2 + c^2\right) \right],$$

$$A_{33} = (1-e^2)^{\frac{1}{2}} p^{-3} e^3 \left[\frac{1}{12} b_1 p^{-1} - \frac{1}{3}\, p^{-2} \left(\frac{1}{2}\, b_2^2 + c^2\right) \right],$$

$$A_{34} = -\frac{1}{32} (1-e^2)^{\frac{1}{2}} p^{-5} e^4 \left(\frac{1}{2} b_2^2 + c^2\right),$$

$$2\pi\nu_1=(-2\alpha_1)^{\frac{1}{2}}(a+b_1+A_1+c^2\eta_0^2A_2B_1B_2^{-1})^{-1},$$

$$2\pi\nu_2=(\alpha_2^2-\alpha_3^2)^{\frac{1}{2}}\eta_0^{-1}A_2B_2^{-1}(a+b_1+A_1+c^2\eta_0^2A_2B_1B_2^{-1})^{-1}.$$

The uniformising variables E, v, and ψ are then given by $E=M_s+E_p$, $v=M_s+v_p$, and $\psi=\psi_s+\psi_p$. If t is the time, their secular parts M_s and ψ_s are given exactly by

$$M_s=2\pi\nu_1(t+\beta_1-c^2\beta_2\alpha_2^{-1}\eta_0^2B_1B_2^{-1}),$$

$$\psi_s=2\pi\nu_2[t+\beta_1+\beta_2\alpha_2^{-1}(a+b_1+A_1)A_2^{-1}].$$

Let the periodic parts be split as follows: $E_p=E_0+E_1+E_2$, $v_p=v_0+v_1+v_2$, and $\psi_p=\psi_0+\psi_1+\psi_2$, where, e.g., E_0 contains terms of order k^0, k, and k^2, E_1 contains terms of order k and k^2, and E_2 contains only terms of order k^2.

Then E_0 is given by the Kepler equation

$$M_s+E_0-e'\sin(M_s+E_0)=M_s,$$

where $e'\equiv a(ae+b_1)^{-1}<e$. The term v_0 is then given by placing $v=M_s+v_0$ and $E=M_s+E_0$ in the *anomaly connections*

$$\cos v=(\cos E-e)(1-e\cos E)^{-1}\qquad \sin v=+(1-e^2)^{\frac{1}{2}}(1-e\cos E)^{-1}\sin E$$

or equivalent relations. (Note that the e here is the original e and not the e' in the Kepler equation.) Then

$$\psi_0=(-2\alpha_1)^{-\frac{1}{2}}(\alpha_2^2-\alpha_3^2)^{\frac{1}{2}}\eta_0^{-1}A_2B_2^{-1}v_0.$$

The term E_1 is now given by

$$E_1=[1-e'\cos(M_s+E_0)]^{-1}M_1-\tfrac{1}{2}e'[1-e'\cos(M_s+E_0)]^{-3}M_1^2\sin(M_s+E_0),$$

where

$$M_1\equiv(a+b_1)^{-1}\left[-(A_1+c^2\eta_0^2A_2B_1B_2^{-1})v_0+\frac{c^2}{4}(-2\alpha_1)^{\frac{1}{2}}(\alpha_2^2-\alpha_3^2)^{-\frac{1}{2}}\eta_0^3\sin(2\psi_s+2\psi_0)\right].$$

The term v_1 is then given by placing $v=M_s+v_0+v_1$ and $E=M_s+E_0+E_1$ in the *anomaly connections*. Then

$$\psi_1=(-2\alpha_1)^{-\frac{1}{2}}(\alpha_2^2-\alpha_3^2)^{\frac{1}{2}}\eta_0^{-1}B_2^{-1}[A_2v_1+A_{21}\sin(M_s+v_0)+A_{22}\sin(2M_s+2v_0)]+\frac{q^2}{8}B_2^{-1}\sin(2\psi_s+2\psi_0)$$

Finally

$$E_2=[1-e'\cos(M_s+E_0+E_1)]^{-1}M_2,$$

where

$$M_2\equiv-(a+b_1)^{-1}\Big[A_1v_1+A_{11}\sin(M_s+v_0)+A_{12}\sin(2M_s+2v_0)$$
$$+c^2(-2\alpha_1)^{\frac{1}{2}}(\alpha_2^2-\alpha_3^2)^{-\frac{1}{2}}\eta_0^3\left\{B_1\psi_1-\frac{1}{2}\psi_1\cos(2\psi_s+2\psi_0)-\frac{q^2}{8}\sin(2\psi_s+2\psi_0)+\frac{q^2}{64}\sin(4\psi_s+4\psi_0)\right\}\Big].$$

Then v_2 is found by placing $v=M_s+v_0+v_1+v_2$ and $E=M_s+E_0+E_1+E_2$ in the *anomaly connections* and

$$\psi_2=(-2\alpha_1)^{-\frac{1}{2}}(\alpha_2^2-\alpha_3^2)^{\frac{1}{2}}\eta_0^{-1}B_2^{-1}[A_2v_2+A_{21}v_1\cos(M_s+v_0)$$
$$+2A_{22}v_1\cos(2M_s+2v_0)+A_{23}\sin(3M_s+3v_0)+A_{24}\sin(4M_s+4v_0)]$$
$$+\frac{q^2}{4}B_2^{-1}\left[\psi_1\cos(2\psi_s+2\psi_0)+\frac{3q^2}{8}\sin(2\psi_s+2\psi_0)-\frac{3q^2}{64}\sin(4\psi_s+4\psi_0)\right].$$

The spheroidal coordinates ρ and η are then given by

$$\rho=a(1-e\cos E)=(1+e\cos v)^{-1}p,\qquad \eta=\eta_0\sin\psi,\text{ where}$$
$$E=M_s+E_0+E_1+E_2,\qquad v=M_s+v_0+v_1+v_2,\qquad \psi=\psi_s+\psi_0+\psi_1+\psi_2.$$

The right ascension is

$$\phi=\beta_3+\alpha_0(\alpha_2^2-\alpha_1^2)^{-\frac{1}{2}}\eta_0\left[(1-\eta_0^2)^{-\frac{1}{2}}(1-\eta_2^{-2})^{-\frac{1}{2}}\chi+B_3\psi+\frac{3}{32}\,\eta_0^2\eta_2^{-4}\sin 2\psi\right]$$
$$-c^3\alpha_3(-2\alpha_1)^{-\frac{1}{2}}\left[A_3v+\sum_{n=1}^{4}A_{3n}\sin nv\right].$$

Here χ is an angle that equals ψ whenever ψ is a multiple of $\pi/2$ and which also satisfies $\tan\chi=(1-\eta_0^2)^{\frac{1}{2}}\tan\psi$. If α_3 is positive or negative, the orbit is respectively direct or retrograde.

The above solution gives secular terms of the intermediary orbit exactly and periodic terms correctly through order k^2. There are no long-period terms.

10. List of Symbols

We here list only the symbols that are most frequently used, giving for each a short definition or the number of the equation in which it first appears. Note that the first digit in each such number is the number of the section.

Symbol — *Definition or equation number*

a—3.23; element of 2d set,
a_0—3.3; element of 1st set,
A—3.11 (also 4.12),
A_1—5.31,
A_2—5.36,
A_{11}—5.32,
A_{12}—5.33,
A_{2j}—5.37 through 5.40 for $j=1, 2, 3, 4$,
A_3—5.61,
A_{3j}—5.62 through 5.65 for $j=1, 2, 3, 4$,
b_1,b_2—5.6,
B—3.11 (also 4.13)
B_1—6.40,
B_2—6.41,
B_3—6.65,
B_{3n}—6.64 (also 6.68)
c—2.4 (fundamental distance in potential and in definitions of coordinates),
d_f—5.46,
D_m—5.44 (also 5.50 and 5.53),
e—3.24; element of 2d set,
e_0—3.4; element of 1st set,
e'—8.32,
E—5.12; "eccentric anomaly,"
E_s—8.17; secular part of $E(=M_s)$,
E_p—8.17; periodic part of E,
E_0,E_1,E_2—various terms of E_p,
$E(q)$—complete elliptic integral of second kind, modulus q,
$E(\psi,q)$—incomplete elliptic integral of second kind; 6.9 and 6.10,
$f(\rho)$—2.10,
$F(\rho)$—2.8,
$F(\psi,q)$—incomplete elliptic integral of the first kind; 6.9 and 6.10,
$G(\eta)$—2.9,
h—5.7,
i_0—3.5; element of 1st set,
I—4.7; element of 2d set,
I_c—value of I for which $\lambda=1$,
J_2—2.4; coefficient of second zonal harmonic of potential,

Symbol — *Definition or equation number*

J_1,J_2,J_3—action variables; (7.1) through (7.3),
J_{mn}—7.8,
k_0—3.16; c^2/p_0^2,
k—4.17; c^2/p^2,
$K(q)$—6.15; complete elliptic integral of first kind, modulus q,
L_0—6.46,
L_m—6.45,
L_{1n}—6.49,
M_s—8.20; secular part of "mean anomaly,"
M_1,M_2—8.37, 8.45; periodic parts of "mean anomaly," of orders k and k^2,
N_1,N_2,N_3—6.1 through 6.3; η-integrals,
p_0—$a_0(1-e_0^2)$,
p—$a(1-e^2)$,
P_n—Legendre polynomial of degree n,
q—η_0/η_2,
r—2.1 and 2.2; geocentric distance,
r_e—equatorial radius,
r_1,r_2—3.1, 3.2; zeros of $f(\rho)$,
R_1,R_2,R_3—5.1 through 5.3; ρ-integrals,
$R_m(z)$—$z^mP_m(z^{-1})$,
sgn α_3—sign of α_3,
t—time,
v—5.12; "true anomaly,"
v_s—8.17; secular part of $v(=M_s)$
v_p—8.17; periodic part of v,
v_0,v_1,v_2—various terms of v_p,
V_e—2.3; the potential of this paper,
X,Y,Z—2.1 and 2.2; rectangular coordinates,
x—3.17; $(1-e_0^2)^{1/2}$,
y—3.18; $\alpha_3/\alpha_2=\cos i_0$,
α_1—total energy, first Jacobi const, <0 for a satellite orbit,
α_2—second Jacobi constant,
α_3—Z-component of angular momentum, third Jacobi constant,
β_1,β_2,β_3—Jacobi β's,
γ—8.1 d,
γ_m—6.66,
δ_n—5.47,
ϵ—6.81,

Symbol	Definition or equation number
η	2.1 and 2.2; a spheroidal coordinate $\rightarrow \sin\theta$ as $r\rightarrow\infty$,
η_0	3.33, 3.36, 3.39; during motion $-1\leq-\eta_0\leq\eta\leq\eta_0\leq1$,
η_2	3.33, 3.37, $\eta_2>>1$,
θ	2.1 and 2.2; geocentric declination,
λ	5.6; b_1/b_2,
ν_1,ν_2,ν_3	7.4 through 7.6,
ρ	2.1 and 2.2; a spheroidal coordinate $\rightarrow r$ as $r\rightarrow\infty$,
ϕ	2.1 and 2.2; geocentric right ascension; the third spheroidal coordinate,
χ	6.51,
ψ	6.4; a uniformizing variable analogous to the argument of latitude,
ψ_s	8.17; secular part of ψ,
ψ_p	8.17; periodic part of ψ,
ψ_0,ψ_1,ψ_2	various terms of ψ_p,
$\bar{\omega}$	mean motion of ρ-perigee relative to the ascending node,
$\bar{\Omega}$	mean motion of the ascending node relative to OX.

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Note on the "Baffled Piston" Problem

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(April 7, 1961)

King's integral expression governing the problem of the time-harmonic motion of a piston in a rigid wall is simplified.

In a previous paper [4][2] the case of an arbitrarily (not necessarily time-harmonic) moving disk in a rigid wall was discussed. The starting point was the eq (12) in the above quoted paper,

$$U=a\int_0^\infty (\tau^2+\gamma^2)^{-\frac{1}{2}}\exp\left[-z(\tau^2+\gamma^2)^{-\frac{1}{2}}\right]J_0(\tau\rho)J_1(\tau a)\,d\tau, \tag{1}$$

which, when γ is replaced by ik $(k=2\pi/\lambda)$ represents the velocity potential for the time harmonic case (King's integral [2]). The investigation of transient phenomena as carried out in [4] did not involve (1) directly, but its inverse Laplace transform with respect to γ, and therefore not much attention was paid to (1) itself. But the computation of the acoustic pressure distribution for the time-harmonic case makes an investigation of (1) itself imperative. Numerous approximate evaluations of (1) (with γ replaced by ik, $k>0$) have been carried out. (for instance, [1], [2], [3], [5], [6]). It may be noted in this connection that (1) can be considerably simplified in as much as that it can be transformed into an integral expression with an elementary integrand which is extended over a finite interval of integration. However, the expressions (22) and (23) in [4] for the transient field can be obtained as easily from this simplified expression for U.

We proceed now to replace the factor $(\tau^2+\gamma^2)^{-\frac{1}{2}}\exp[-z(\tau^2+\gamma^2)^{\frac{1}{2}}]$ by the integral formula in [4(9)]:

$$(\tau^2+\gamma^2)^{-\frac{1}{2}}\exp\left[-z(\tau^2+\gamma^2)^{\frac{1}{2}}\right]=\int_0^\infty J_0(\tau t)\,t(t^2+z^2)^{-\frac{1}{2}}\exp\left[-\gamma(t^2+z^2)^{\frac{1}{2}}\right]dt.$$

Upon interchanging the order of integration, one obtains instead of (1)

$$U=a\int_0^\infty t(t^2+z^2)^{-\frac{1}{2}}\exp\left[-\gamma(t^2+z^2)^{\frac{1}{2}}\right]\left\{\int_0^\infty J_0(\tau\rho)J_0(\tau t)J_1(\tau a)\,d\tau\right\}dt.$$

The inner integral is known [4(20)] and therefore, instead of (1),

$$U=a\int_0^\infty t(t^2+z^2)^{-\frac{1}{2}}\exp\left[-\gamma(t^2+z^2)^{\frac{1}{2}}\right]f(t)\,dt, \tag{2}$$

where

$$f(t)=\begin{cases}0, & 0<a<|t-\rho|\\ (\pi a)^{-1}\arccos\dfrac{t^2+\rho^2-a^2}{2\rho t}, & |t-\rho|<a<t+\rho\\ a^{-1}, & a>t+\rho\end{cases}$$

Again, as before, two cases A and B have to be distinguished. The results are

Case A, $a>\rho$:

$$f(t)=\begin{cases}a^{-1} & 0<t<a-\rho\\ (\pi a)^{-1}\arccos\dfrac{t^2+\rho^2-a^2}{2\rho t} & a-\rho<t<a+\rho\\ 0 & t>a+\rho\end{cases}$$

Case B, $a<\rho$:

$$f(t)=(\pi a)^{-1}\arccos\frac{t^2+\rho^2-a^2}{2\rho t}\qquad \rho-a<t<\rho+a$$
$$=0\qquad \text{otherwise.}$$

Therefore, inserting $f(t)$ into (2),

Case A, $\rho<a$:

$$U=\int_0^{a-\rho} t(t^2+z^2)^{-1/2}\exp[-\gamma(t^2+z^2)^{1/2}]dt+\pi^{-1}\int_{a-\rho}^{a+\rho} t(t^2+z^2)^{-1/2}\exp[-\gamma(t^2+z^2)^{1/2}]\arccos\left(\frac{t^2+\rho^2-a^2}{2\rho t}\right)dt.$$

or, upon evaluating the first integral,

$$U=\frac{1}{\gamma}\exp(-\gamma z)-\frac{1}{\gamma}\exp\{-\gamma[z^2+(a-\rho)^2]^{1/2}\}+\pi^{-1}\int_{a-\rho}^{a+\rho} t(t^2+z^2)^{-1/2}\exp[-\gamma(t^2+z^2)^{1/2}]\arccos\left(\frac{t^2+\rho^2-a^2}{2\rho t}\right)dt.$$

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[2] Figures in brackets indicate the literature references at the end of this paper.

Case B, $\rho > a$:

$$U=\pi^{-1}\int_{\rho-a}^{\rho+a} t(t^2+z^2)^{-1/2} \exp[-\gamma(t^2+z^2)^{1/2}] \arccos\left(\frac{t^2+\rho^2-a^2}{2\rho t}\right)dt.$$

Substitute $t^2+z^2=\alpha^2$ and get

$$U=\frac{1}{\gamma}\exp(-\gamma z)-\frac{1}{\gamma}\exp\{-\gamma[z^2+(a-\rho)^2]^{1/2}\}+\pi^{-1}\int_{R'}^{R} \exp(-\gamma\alpha) \arccos\left[\frac{\alpha^2-z^2+\rho^2-a^2}{2\rho(\alpha^2-z^2)^{1/2}}\right]d\alpha, \qquad \rho<a, \quad (3)$$

$$U=\pi^{-1}\int_{R'}^{R} \exp(-\gamma\alpha) \arccos\left[\frac{\alpha^2-z^2+\rho^2-a^2}{2\rho(\alpha^2-z^2)^{1/2}}\right]d\alpha, \qquad \rho>a. \quad (4)$$

As in [4(21)], R and R' denote the smallest and the largest distance of a point of observation from the circumference of the disk respectively. These properties are given by

$$R=[z^2+(a+\rho)^2]^{1/2}, \qquad R'=[z^2+(a-\rho)^2]^{1/2}.$$

The expressions (3) and (4) for U are considerably simpler than [4(12)]. For the time-harmonic case, γ has to be replaced by ik. The expressions [4(22)] and [4(23)] for a "Dirac pulse" are immediately obtained from (3) and (4) by taking their Laplace inversions with respect to γ.

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Some Results on Non-Negative Matrices

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(March 31, 1961)

Recently Mirsky and Farahat proposed the problem of characterizing the class of doubly stochastic matrices for which the least number of permutation matrices necessary to represent it as a convex sum has a prescribed value. It is shown that this number can be related to the number of eigenvalues of modulus one. The problem of similarity of doubly stochastic matrices is also treated. Finally, the question of transitivity of powers of sets of functions on the first n positive integers into itself is treated by defining a corresponding incidence matrix and examining its powers.

I. Introduction

In 1946 [1][3] Birkhoff proved that any doubly stochastic (d.s.) matrix with non-negative entries is a convex combination of permutation matrices. We recall that a matrix is d.s. if every row and column sum is 1. Let Ω_n denote the polyhedron of non-negative d.s. matrices and as noted in [5] dim $\Omega_n = (n-1)^2$ and hence any $A \epsilon \Omega_n$ can be written as a convex combination of at most $(n-1)^2+1$ permutation matrices. Mirsky and Farahat in a recent paper [6] suggested an investigation of the minimum number $\beta(A)$ of permutation matrices necessary to represent $A \epsilon \Omega_n$ as a convex combination. In section 2 we obtain an inequality relating $\beta(A)$ to $h(A)$, the number of characteristic roots of A of absolute value 1. In section 2 we also study a problem of the similarity of two matrices in Ω_n.

In 3 we discuss conditions on $A \geq 0$ that are implied by the equality $\det (I-A) = \prod_{i=1}^{n} (1-a_{ii})$.

In 4 we obtain an inequality for the difference between the maximal characteristic roots of two matrices A and B satisfying $B \geq A \geq 0$, where $B \geq A$ means $b_{ij} \geq a_{ij}$ for all i,j. In certain cases our inequality yields better bounds than those obtained by applying the results in [2] or [7].

In 5 we discuss the question of transitivity of sets of functions on the set of integers $\{1, \ldots, n\}$ into itself by matrix methods.

2. Doubly Stochastic Matrices

A precise statement of the result relating $h(A)$ and $\beta(A)$ is contained in theorem 3. We prove two preliminary results of some interest in themselves. Recall that $A \geq 0$ is *reducible* if there exists a permutation matrix P such that

$$PAP' = \begin{pmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{pmatrix}, \tag{1}$$

where A_{11} and A_{22} are square matrices. It is clear that if $A \epsilon \Omega_n$ then (1) implies that $A_{21}=0$. It follows that in case $A \epsilon \Omega_n$ a permutation matrix P may be chosen so that PAP' is the direct sum of irreducible d.s. matrices and 1-square matrices.

THEOREM 1. *If $S \epsilon \Omega_n$ and S is irreducible, then $h(S)$ is a divisor of n.*

PROOF. By the classical Perron-Frobenius theorem, more recently reproved in [8], there exists a permutation matrix P such that

$$PSP' = \begin{pmatrix} 0_1 & S_1 & 0 & \ldots & 0 \\ . & 0_2 & S_2 & \ldots & 0 \\ . & & & & \\ . & & & & S_{h-1} \\ S_h & 0 & & \ldots & 0_h \end{pmatrix}, \tag{2}$$

where 0_i is a principal square matrix of zeros and $h(S)=h$. Let $r(X)$ $(c(X))$ denote the number of rows (columns) of X. Clearly since $S \epsilon \Omega_n$ each S_i is d.s. and $r(S_i)$ is the sum of the elements of S_i as is $c(S_i)$. Hence,

$$r(S_i)=c(S_i)=c(0_{i+1})=r(0_{i+1})=r(S_{i+1})=c(S_{i+1}). \tag{3}$$

Thus every S_i is of the same order and $h(S)$ is a divisor of n. We next obtain a relation between $\beta(S)$ and $\beta(S_i)$ where $S=S_1 \dotplus \ldots \dotplus S_m$ and $\dotplus$ denotes the direct sum.

THEOREM 2. *If $S_i \epsilon \Omega_{n_i}$, $i=1, \ldots, m$, $n=\sum_{i=1}^{m} n_i$ and $S=S_1 \dotplus \ldots \dotplus S_m$, then*

$$\beta(S) \leq \sum_{i=1}^{m} \beta(S_i) - m + 1. \tag{4}$$

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[2] The work of this author was supported in part by the U.S. Office of Naval Research.
[3] Figures in brackets indicate the literature references at the end of this paper.

PROOF. First we prove by induction on $r+s$ that if

$$A=\sum_{i=1}^{r}\theta_i P_i, \qquad B=\sum_{j=1}^{s}\varphi_j Q_j,$$

$0<\theta_1\leq\theta_2\leq\ldots\leq\theta_r, 0<\varphi_1\leq\varphi_2\ldots\leq\varphi_s, \sum_{i=1}^{r}\theta_i=1=\sum_{j=1}^{s}\varphi_j$, P_i, Q_j are permutation matrices, $i=1,\ldots,r$, $j=1,\ldots,s$, then $A\dot{+}B$ is in the convex hull of no more than $r+s-1$ permutation matrices of the form $P_i\dot{+}Q_j$. When $r+s=2$, then $A=P_1$, $B=Q_1$, and $A\dot{+}B=P_1\dot{+}Q_1$. Suppose for convenience that $\phi_1\leq\phi_1$. Then

$$A\dot{+}B=\theta_1(P_1\dot{+}Q_1)+(1-\theta_1)\left[\left(\sum_{i=2}^{r}\frac{\theta_i}{1-\theta_1}P_i\right)\right.$$
$$\left.\dot{+}\left(\frac{\varphi_1-\theta_1}{1-\theta_1}Q_1+\sum_{j=2}^{s}\frac{\varphi_j}{1-\theta_1}Q_j\right)\right], \quad (5)$$

and $\sum_{j=2}^{r}\frac{\theta_j}{1-\theta_1}=1$, and

$$\frac{\varphi_1-\theta_1}{1-\theta_1}+\sum_{j=2}^{s}\frac{\varphi_j}{1-\theta_1}=\frac{1}{1-\theta_1}\sum_{j=1}^{s}\varphi_j-\frac{\theta_1}{1-\theta_1}=1.$$

Thus $A\dot{+}B=\theta_1(P_1\dot{+}Q_1)+(1-\theta_1)R$ where $R\epsilon\Omega_n$ is the direct sum of two d.s. matrices, the first involving $r-1$ permutation matrices and the second involving s permutation matrices. We apply the induction to R to conclude that $\beta(R)\leq r+s-2$. Hence $\beta(A\dot{+}B)\leq r+s-1$. Then an obvious induction on m yields

$$\beta(S)=\beta(S_1\dot{+}\ldots\dot{+}S_m)$$
$$\leq\beta(S_1\dot{+}\ldots\dot{+}S_{m-1})+\beta(S_m)-1$$
$$\leq\sum_{j=1}^{m-1}\beta(S_j)-m+2+\beta(S_m)-1$$
$$=\sum_{j=1}^{m}\beta(S_j)-m+1.$$

THEOREM 3. *If $S\epsilon\Omega_n$ and S is irreducible, then*

$$\beta(S)\leq h\left(\frac{n}{h}-1\right)^2+1. \quad (6)$$

where $h=h(S)$.

PROOF. We first observe that if P and Q are permutation matrices then $\beta(PSQ)=\beta(S)$ for any $S\epsilon\Omega_n$. By theorem 1 let $n=qh$ where q is an integer and let R be the permutation matrix with 1 in positions

$$(q+i,\ i) \mod n.$$

Then from (2) we easily compute that

$$PSP'R=S_1\dot{+}\ldots\dot{+}S_{h-1}\dot{+}S_h, \quad (7)$$

and hence by theorem 2

$$\beta(PSP'R)=\beta(S)\leq\sum_{i=1}^{h}\beta(S_i)-h+1.$$

But by the Birkhoff result quoted in the introduction we have that

$$\beta(S_i)\leq(q-1)^2+1.$$

Hence,

$$\beta(S)\leq h[(q-1)^2+1]-h+1=h\left(\frac{n}{h}-1\right)^2+1.$$

Actually the above techniques may be applied to yield better estimates on $\beta(S)$ than that given by a direct application of (6). There are several alternatives for each of the matrices S_i in (7): S_i is reducible; S_i is irreducible with $h(S_i)>1$; S_i is irreducible with $h(S_i)=1$; i.e., S_i is *primitive*. These situations are covered by the following two theorems.

THEOREM 4. *If $S\epsilon\Omega_n$ and P is such a permutation matrix that $PSP'=T_1\dot{+}\ldots\dot{+}T_k$ where each T_j is an n_j-square matrix either irreducible or with $n_j=1$, then*

$$\beta(S)\leq\sum_{j=1}^{k}\left(\frac{n_j}{h(T_j)}-1\right)^2h(T_j)+1.$$

THEOREM 5. *If $S\epsilon\Omega_n$ and P and Q are such permutation matrices that*

$$PSQ=S_1\dot{+}\ldots\dot{+}S_m, \quad (8)$$

where each S_j is an n_j-square matrix either primitive or with $n_j=1$, then

$$\beta(S)\leq\sum_{j=1}^{m}(n_j-1)^2+1.$$

We describe the procedure for obtaining the form (8) for S: First reduce S to the direct sum of irreducible or 1-square d.s. matrices. Each of these may then be reduced to the form (2) which in turn is reduced to a direct sum of irreducible or 1-square matrices. The process is repeated until every matrix which appears is either 1-square or primitive. We illustrate this with the following example: Let $S\epsilon\Omega_8$ be given by

$$S=\begin{pmatrix}0_4 & J_4\\ A & 0_4\end{pmatrix},$$

where $A=\begin{pmatrix}0_2 & I_2\\ J_2 & 0_2\end{pmatrix}$, 0_t and I_t are t-square zero and identity matrices respectively, and $J_t\epsilon\Omega_t$ is the t-square matrix all of whose entries are $1/t$. We verify directly that no coordinate subspace is held invariant by S and hence S is irreducible. We first note that $S^2=J_4\dot{+}J_4$ and hence the characteristic roots of S^2 are 1 and 0. Hence the characteristic

roots of S are 1, -1, and 0 with appropriate multiplicities. On the other hand, 1 is simple by the Perron-Frobenius result and $\mathrm{tr}(S)=0$. Hence the characteristic roots of S must be 1, -1, and 0 (six times) and $h(S)=2$. Thus S is already in the form (2) and we bring S by a permutation to the form $J_4 \dot{+} A$ and by a further one-sided permutation to $J_4 \dot{+} I_2 \dot{+} J_2 = J_4 \dot{+} I_1 \dot{+} I_1 \dot{+} J_2$.

Thus by theorem 2,

$$\beta(S) \leq \beta(J_4)+\beta(I_1)+\beta(I_1)+\beta(J_2)-4+1. \tag{9}$$

A direct application of theorem 3 to J_4 yields $\beta(J_4) \leq 10$ and (9) implies that

$$\beta(S) \leq 11. \tag{10}$$

Of course it is obvious by inspection that $\beta(J_4)=4$ and hence (9) gives $\beta(S) \leq 5$. Note that a direct application of theorem 3 yields $\beta(S) \leq 19$ and the bound given by the dimension of Ω_8 is $\beta(S) \leq 50$. As a matter of fact, $\beta(S)=4$.

Recently one of the present authors and A. J. Hoffman independently proved that if $S \epsilon \Omega_n$ then there exists a permutation σ such that $\prod_{i=1}^{n} s_{i\sigma(i)} \geq 1/n^n$. It follows that the permanent of S satisfies the following inequality:

$$\text{per } (S) = \sum_{\sigma} \prod_{i=1}^{n} s_{i\sigma(i)} \geq 1/n^n$$

for $S \epsilon \Omega_n$. Since

$$S = \sum_{i=1}^{\beta(S)} \alpha_i P_i,$$

where P_i are permutation matrices and $\alpha_i \geq 0$, $\sum_{i=1}^{\beta(S)} \alpha_i = 1$, we see that

$$\text{per } (S) \geq (1/\beta(S))^{n-1}.$$

It follows that we may apply the estimates of theorems 3, 4, 5 to obtain a lower bound for S that in some cases is better than $1/n^n$. These results are motivated by a conjecture of v. der Waerden [4] concerning the minimum value of the permanent of a matrix in Ω_n.

We next discuss circumstances under which two matrices in Ω_n are similar. If $S \epsilon \Omega_n$ and $XSX^{-1} \epsilon \Omega_n$ it of course is false that either X or X^{-1} must be d.s.; e.g., take $S=I_n$. However if the root 1 of S is simple then we have the following result.

THEOREM 6. *If $S \epsilon \Omega_n$ and S is irreducible and $XSX^{-1}=R \epsilon \Omega_n$, then X is a multiple of a d.s. matrix. Moreover, there exists $Y \epsilon \Omega_n$ such that $YSY^{-1}=R$.*

PROOF. Let J be the n-square matrix all of whose entries are 1, and let $\sigma_i(X)$ be the sum of the entries in the ith row of X. Then $XS=RX$ implies that $XSJ=RXJ$, $XJ=R(XJ)$. Each column of XJ is equal to the n-tuple $(\sigma_1(X), \ldots, \sigma_n(X)) = u(X)$. Then $Ru(X)=u(X)$ and since 1 is a simple characteristic root of S and R (S is irreducible), $u(X)$ must be a scalar multiple of the vector e all of whose coordinates are 1. Thus $\sigma_i(X)=\alpha$, $i=1, \ldots, n$, and by a similar argument we easily see that $\sigma_i(X')=\beta$, $i=1, \ldots, n$. But then $\alpha=\beta$ and X is a scalar multiple of a d.s. matrix. Now X commutes with J and thus the characteristic roots of $X+kJ$ are λ_1+kn, $\lambda_2, \ldots, \lambda_n$, where λ_i are the roots of X and λ_1 corresponds to the characteristic vector e. Choose k so that $\lambda_1+kn \neq 0$ and $X+kJ>0$, and let $Y=(\alpha+nk)^{-1}(X+kJ)$. This completes the proof.

3. Triangular Non-Negative Matrices

THEOREM 7. *Let $A \geq 0$ with maximal non-negative characteristic root $r \leq 1$. Then a necessary and sufficient condition that there exists a permutation matrix P such that PAP' is triangular is*

$$\prod_{i=1}^{n} (1-a_{ii}) = \det (I_n - A). \tag{11}$$

PROOF. We prove first that if

$$f(t) = \prod_{i=1}^{n} (t-a_{ii}) - \det (tI_n - A), \tag{12}$$

then $f(t) \geq 0$ for $t \geq r$. In this argument and subsequently we use the classical result that states that if α is the dominant non-negative characteristic root of a principal submatrix of A, then $\alpha \leq r$ and if A is irreducible then the inequality is strict. Hence, $f(r)=\prod_{i=1}^{n}(r-a_{ii}) - \det (rI_n - A) \geq 0$ and the inequality is strict if A is irreducible. To prove that $f(t) \geq 0$ we proceed by induction on n. Now differentiating (12) we have

$$f'(t) = \sum_{i=1}^{n} \left\{ \prod_{j \neq i} (t-a_{jj}) - \det (tI_{n-1} - A(i|i)) \right\}, \tag{13}$$

where $A(i|i)$ is the principal submatrix of A obtained by omitting the ith row and column. For each i the summand in (13) is non-negative by the induction hypothesis (since $t \geq r$ and r is not smaller than the dominant non-negative characteristic root of $A(i|i)$). Thus $f'(t) \geq 0$ and $f(r) \geq 0$ so that $f(1) \geq 0$. Moreover if A were irreducible it would follow that $f(1) \geq f(r) = \prod_{i=1}^{n}(r-a_{ii}) > 0$.

We complete the proof as follows. Choose a permutation matrix P such that PAP' has the following sequence of blocks down the main diagonal.

$$B_1 \dot{+} \ldots \dot{+} B_m = B,$$

where each B_i is 1-square or possibly n_i-square irreducible. We show that the second alternative is impossible and it follows that PAP' is triangular. Clearly (11) implies that

$$\prod_{i=1}^{n} (1-a_{ii}) = \prod_{i=1}^{n} (1-b_{ii}) = \det (I_n - A) = \det (I_n - B)$$

$$= \prod_{k=1}^{m} \det (I_{n_k} - B_k).$$

Suppose some $B_k=C$ is irreducible. Then by the above argument

$$\prod_{i=1}^{n_k}(1-c_{ii})>\det(I_{n_k}-B_k),$$

and we would conclude that

$$\prod_{k=1}^{m}\det(I_{n_k}-B_k)<\prod_{i=1}^{n}(1-a_{ii}),$$

a contradiction. This completes the proof.

4. Inequalities for the Dominant Characteristic Root

Let $E^{(n)}$ denote the unit n-simplex. If $A\geq 0$ is irreducible, let α be the dominant positive root and let $0<x\epsilon E^{(n)}$ be the unique characteristic vector of A in $E^{(n)}$ corresponding to α. Suppose $R\geq 0$, $R\neq 0$ commutes with A. Then $Ax=\alpha x$, $RAx=\alpha Rx$, $A(Rx/\sigma(Rx))=\alpha(Rx/\sigma(Rx))$ where if $y=(y_1\ \ldots\ y_n)$ then $\sigma(y)=\sum_{i=1}^{n}y_i$. But then the uniqueness of x implies that $Rx/\sigma(Rx)=x$ because $(Rx/\sigma(Rx))\epsilon E^{(n)}$.

THEOREM 8. *Let $B\geq A$ be two non-negative irreducible n-square matrices with dominant positive roots β and α respectively. Let $M=\max_{i,j}(b_{ij}-a_{ij})$, $m=\min_{i,j}(b_{ij}-a_{ij})$ and suppose that R is a non-negative matrix which commutes with either A or B and possesses no zero columns. Then*

$$\frac{m}{\max_i \max_j \frac{r_{ij}}{\sum_{k=1}^{n}r_{kj}}}\leq\beta-\alpha\leq\frac{M}{\min_i \min_j \frac{r_{ij}}{\sum_{k=1}^{n}r_{kj}}}. \tag{14}$$

PROOF. Let $y\epsilon E^{(n)}$, $y>0$, satisfy $Ay=\alpha y$ and let $\mu=(By)_i/y_i=\max_k(By)_k/y_k\geq\beta$. Then we assert that $\beta-\alpha\leq M/y_i$; for $(B-A)y+Ay=By$,

$$(B-A)y+\alpha y=By,$$

$$\sum_{k=1}^{n}(b_{ik}-a_{ik})y_k+\alpha y_i=(By)_i,$$

$$\frac{1}{y_i}\sum_{k=1}^{n}(b_{ik}-a_{ik})y_k+\alpha=\frac{(By)_i}{y_i}=\mu\geq\beta.$$

Now $y\epsilon E^{(n)}$; hence $\sum_{k=1}^{n}(b_{ik}-a_{ik})y_k\leq M$, and $\beta-\alpha\leq M/y_i\leq M/\min_k y_k$.

From the above discussion we know that $y=Ry/\sigma(Ry)$. Consider the function

$$z_k(x)=\frac{\sum_{j=1}^{n}r_{kj}x_j}{\sum_{j=1}^{n}\sum_{i=1}^{n}r_{ij}x_j};\qquad x\epsilon E^{(n)}; \tag{15}$$

minimizing the right-hand side of (15) over all $x\epsilon E^{(n)}$ yields

$$y_k\geq\min_i\frac{r_{ki}}{\sum_{i=1}^{n}r_{ii}}.$$

Thus

$$\beta-\alpha\leq M/\min_k y_k\leq M/\min_k\min_i\frac{r_{ki}}{\sum_{i=1}^{n}r_{ii}}.$$

A similar computation establishes the lower bound in (14).

Note that R is *any* non-negative matrix commuting with A or B; e.g., any non-negative polynomial in A with no zero columns. We also remark that in practice we compute each column sum of R, divide it into the least element in the column, and take the least such quotient to obtain the denominator of the upper bound in (14). Similarly for the lower bound.

5. A Matrix Condition for Transitivity

Let L_n denote the set of integers $\{1, \ldots, n\}$ and suppose $\sigma_1, \ldots, \sigma_k$ are functions mapping L_n into itself. We define $\mathfrak{A}_k$ to be the set of functions of the form $\sigma_{i_1}\sigma_{i_2}\ldots\sigma_{i_k}$, $1\leq i_j\leq k$. We say that $\mathfrak{A}_p$ is transitive for some p if for any given s and t in L_n there exists $\phi\epsilon\mathfrak{A}_p$ such that $\phi(s)=t$. We define an n-square *incidence matrix* A associated with $\mathfrak{A}_1$ as follows: For each σ_i we define Q_i as the matrix whose (s, t) entry is 1 if $\sigma_i(s)=t$, 0 otherwise; then let $A=\frac{1}{k}\sum_{i=1}^{k}Q_i$. It is clear that $A\geq 0$ and every row sum of A is 1; i.e., A is row stochastic. We remark that if Γ_n denotes the totality of non-negative row stochastic n-square matrices, then an easy argument [6] thows that Γ_n is the convex hull of the set of all matrices with exactly one 1 in each row.

In general let A correspond to $\sigma_1, \ldots, \sigma_k$ and B to $\phi_1, \ldots, \phi_m$. Then the (i, j) entry of AB is positive if and only if the integer i is mapped by some σ_α into p and p is mapped by some ϕ_β into j. That is, $\phi_\beta\sigma_\alpha(i)=j$. Thus we have

THEOREM 9. *A necessary and sufficient condition that $\mathfrak{A}_p$ be transitive is that $A^p>0$.*

We easily obtain two corollaries to this result.

COROLLARY 1. *Let $\mathfrak{A}_1=\{\sigma_1, \ldots, \sigma_k\}$ be a set of functions on $L_n=\{1, \ldots, n\}$ into itself satisfying*

(a) *at least one σ_i has a fixed point,*

(b) *no proper subset of L_n is left invariant (as a set) by every $\sigma_i\epsilon\mathfrak{A}_1$. Then $\mathfrak{A}_p$ is transitive forp $\geq 2n-2$.*

PROOF. Condition (b) implies that the incidence matrix A is irreducible. Condition (a) implies that $\mathrm{tr}(A)>0$ and hence $h(A)=1$. Hence A is primitive and the integer $2n-2$ follows from a result of Holladay and Varga [3].

COROLLARY 2. *Let* $\mathfrak{A}_1=\{\sigma_1, \ldots, \sigma_k\}$ *be a set of permutations of the integers* $\{1, \ldots, n\}=L_n$ *satisfying*

(a) *there exists an integer* s *in* L_n *and a pair of permutations* σ_i *and* σ_j *such that* $\sigma_j\sigma_i(s)=s$,

(b) *no proper subset of* L_n *is left invariant (as a set) by every* $\sigma_i\epsilon\mathfrak{A}_1$,

(c) n *is odd.*

Then $\mathfrak{A}_p$ *is transitive for* $p\geq 4n-4$.

PROOF. If there exists a σ_m and an integer v such that $\sigma_m(v)=v$ then we have the conditions of corollary 1 and $\mathfrak{A}_p$ is transitive for $p\geq 2n-2$ without the condition (c). If not, let A be the incidence matrix and then $\mathrm{tr}(A)=0$, but the sum of all the second order principal subdeterminants of A is not zero. Hence $h(A)$ is 1 or 2. But since $A\epsilon\Omega_n$ we apply theorem 1 to conclude that $h(A)$ divides n. Thus $h(A)=1$ and A is primitive. Now A^2 has a positive element on the main diagonal and $h(A^2)=1$. Thus A^2 is also irreducible and thus primitive so that applying the result in [3] again we have $(A^2)^{2n-2}>0$. completing the proof.

Note that condition (c) is essential as shown by the following example: $\mathfrak{A}_1=\{\sigma_1, \sigma_2\}$ where $\sigma_1=(1\,2\,3\,4)$, $\sigma_2=(1\,4\,3\,2)$ satisfies conditions (a) and (b) of corollary 2. However, $\mathfrak{A}_p$ is not transitive for any p.

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Probability Inequalities of the Tchebycheff Type

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Thirteen basic inequalities relating tail area probabilities to moments are stated. One-tailed and multidimensional inequalities as well as the classical two-tailed, unidimensional inequalities are presented. Sufficient detail is given for each inequality so that the material can be used in handbook style without cross referencing or familiarity with the entire article. Examples of uses of the inequalities, tables comparing the relative strengths of the inequalities, and bibliographic data through 1960 are included.

I. Introduction

1. Background

The author originally prepared this survey and bibliography of probability inequalities of the Tchebycheff type, early in 1952, for convenient reference within the Statistical Engineering Laboratory of the National Bureau of Standards. Copies of the bibliography and of the survey were given limited circulation, within the Bureau and among a few individual specialists in this area, in April and June 1952, respectively, with a view to revision for ultimate publication.

Shortly thereafter the author, in his capacity as Associate Editor of the *Journal of the American Statistical Association*, learned of the preparation by H. J. Godwin, University College of Swansea, Wales, of a much more comprehensive treatment of inequalities of the Tchebycheff type, and gave his full support to this undertaking, which resulted in a definitive publication on the subject [Godwin (1955)].[2]

As the years have passed, however, many persons who have had access to the present author's original survey have found it to be a far more convenient source of directly applicable information for practical application than Godwin's more comprehensive paper. Consequently, in response to the urgings of various professional colleagues, the author has brought together and combined in the present paper his 1952 survey and bibliography of probability inequalities of the Tchebycheff type, with a few additions and revisions necessitated by the passage of time and the author's increased knowledge of the subject.

2. Scope and Organization

Tchebycheff inequalities give bounds for the probability of certain events. In particular they give estimates for deviations from the mean in terms of the moments.

A selected collection of Tchebycheff inequalities is given. They have been selected for their diverse nature and for their usefulness in applied and theoretical work.

In section II the various inequalities are presented with some notes on their uses and the conditions under which they may be used. In several cases more than one form of the inequality is presented in order to make it easier to work with the inequality. With each inequality the nature of the random variable is specified; that is, it is indicated whether the random variable is arbitrary, a sum, or some other function. Other special conditions are stated, always including the dimension of the random variables, and the moments which are assumed to exist.

In section III, tables are given for finding the particular probabilities associated with a specific inequality. Tables are also given for determining the required size of the variable parameters when a particular probability is desired. These tables will be found useful in choosing which of the inequalities to use. Tables have not been prepared for some of the inequalities which involve several parameters.

Several examples are given in section IV showing how some of these inequalities can be used. These examples show various possible uses, but are by no means exhaustive.

The bibliography in section V is as complete as possible. The books by Uspensky (1937) and Fréchet (1950), and the paper by Godwin (1955) are recommended as good surveys of the subject.

Tchebycheff inequalities are useful for working with distributions whose functional form is unknown. In many situations it is possible to avoid the assumption that random variables are (say) normally distributed. All that is needed for use of these inequalities are good estimates of certain population moments. Sometimes something is required of the functional form of the density function of a distribution. This is true for inequalities **9, 10,** and **11a** in the text. However, it is easy to verify

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[2] Names followed by dates in parentheses refer to the bibliography at the end of this paper.

whether the necessary conditions are true for the distributions that one is discussing.

In statistical work these inequalities have had several uses. In working theoretical problems, it is often necessary to use these inequalities, for instance, in proving the weak law of large numbers for binomial distributions. These inequalities are particularly useful for testing hypotheses and finding confidence intervals for the mean of a distribution, if one has some information about the other moments. In industrial work, these inequalities have been used to form "tolerance" sets.

Usually one does not have the true values for all of the parameters that are needed for using these inequalities. But if one has upper bounds for the parameters, that is for the moments, then one can use these inequalities. If one has run a process many times with the same type of material, then one usually has a good idea of the variance, even if the process mean has been shifted, so that in a sense one often knows some of the moments, and in this way one can test for the others.

Remember that a sample acts as a population; therefore, once the moments have been computed for a sample, all of these inequalities will be true for that sample; that is, these inequalities will provide bounds for the portion of the sample in various parts of its range. To obtain lower bounds for the probabilities in the tails of the distribution it is usually necessary to assume that the random variables are bounded.

Most of the inequalities presented are for the univariate case. There are several papers that discuss the multivariate case in much more detail; in particular, see Camp (1948), Leser (1942), Pearson (1919), and Marshall and Olkin (1960a). Most of the multivariate inequalities have been omitted because they are quite complicated and hard to apply. For each inequality presented here, the dimension of the random variable is specified, and this is a clue to deciding which one of the inequalities is applicable to a specific problem.

Several of the inequalities given require special assumptions on the shape of the involved distributions. All of these special assumptions require that the distribution has an unique mode. Narumi (1923) treated the opposite case, where the distributions have an unique minimum and increase as you go away from it. This case did not seem to be as important as the other and is omitted.

Winsten (1946) found inequalities that involve the ranges for various sample sizes. These inequalities will undoubtedly prove useful in the future; but they are not entirely analogous to the Tchebycheff inequalities and were omitted.

The Markov Inequality **1**[2] contains many of the other ones as special cases, which is a little surprising since this is the simplest of all the inequalities. This results from the fact that **1** is true for any positive random variable, X, that has a finite expected value. In particular, **1a** is derived from **1** by replacing X by a sum of random variables. Inequality **2** is obtained by replacing X by the square of the difference between a random variable and its expected value. One can derive many of the other inequalities in this manner.

In cases where the inequality is given only for the random variable X minus its mean, there are also inequalities for a sample average minus the expected value of that average.

Most of the inequalities are stated in the form of upper bounds for the probability that a random variable is greater than or equal to some number. There are opposite inequalities, lower bounds for the probability that the random variable is less than the same number. These are the same expressions, with the inequality reversed within the probability symbol (the "greater than or equal" symbol being replaced by a "less than" symbol), and with the right-hand side replaced by one minus the original right-hand side.

As given, some of the inequalities are very weak, for the right-hand sides may be greater than unity; but a probability is always less than or equal to unity, so the right-hand sides should be interpreted as the minimum of the given expression and unity.

Most of these inequalities cannot be improved; that is, the right-hand sides cannot be replaced by smaller quantities. That is, usually the left-hand side equals the right-hand side for some distribution that satisfies the conditions under which the inequality holds. Of course this will only occur for certain exceptional cases. If the exceptional case is known to be impossible, there might be a better inequality available [see Godwin (1955)].

In the following inequalities, unless otherwise noted, λ is any positive number. EX equals the expected value of the random variable, and will be denoted by μ; if need be this will be given a subscript. The expectation sign E will be used to denote other expected values depending on the argument that follows it. That is, $E(X-\mu)^2$ will be the variance and will be denoted by σ^2. The symbol $P(A)$ means the probability of the event A.

It gives me great pleasure to thank the staff of the Statistical Engineering Laboratory of the National Bureau of Standards for their encouragement during the preparation of this manuscript. I am particularly indebted to Joan R. Rosenblatt for her careful reading and detailed comments on the penultimate manuscript and to Lola S. Deming for her skillful management in revising this manuscript for publication. Professor Ingram Olkin of the University of Minnesota has been a guide in removing obscurisms, in suggesting crucial changes to bring the original manuscript up to date, and in the preparation of the final bibliography.

[2] Each inequality has been given a name, mostly for convenience. These names do not necessarily reflect priority. It is hoped that the names do not conflict with common usage.

II. Inequalities

1. Markov

$$P\left(\frac{X}{\mu}\geq\lambda\right)\leq 1/\lambda \tag{1}$$

$$P(X\geq\lambda)\leq\mu/\lambda \tag{1'}$$

Random variable: X

Restrictions: X is non-negative, that is, $P(X<0)=0$

Dimension: One

Moments: $\mu=EX$

References: Cramér (1946), Fréchet (1937).

Notes:

1. This is a fundamental inequality from which inequalities **1a**, **2**, **2a**, **2b**, **3**, **5**, **6**, **7**, **8**, **13**, and **13a** can be derived by picking X as a special function of the random variables of interest.

2. In using this inequality, note that one needs to know only one moment, or one is testing an hypothesis about only one moment.

3. By itself this is a rather weak inequality, for the probability is bounded by $1/\lambda$; this is of course to be expected, since only one moment is being used and therefore one has very little knowledge about the involved distribution, or at least is only using very little of this knowledge.

4. Equality holds in **1**, with $\lambda>1$, for the random variable X which takes only the values zero and $\mu\lambda$, with probabilities $(\lambda-1)/\lambda$ and $1/\lambda$, respectively. If $\lambda<1$, the left-hand side of **1** is unity for the degenerate random variable which is equal to μ with probability one. A similar argument shows that **1'** is also a sharp inequality.

1a. Markov

$$P\left(\frac{X}{\mu}\geq\lambda\right)\leq 1/\lambda \tag{1a}$$

$$P\left(\sum_{i=1}^{n}X_i\Big/\sum_{i=1}^{n}\mu_i\geq\lambda\right)\leq 1/\lambda \tag{1a'}$$

Random variable:

$$X=\sum_{i=1}^{n}X_i$$

Restrictions: Each X_i is non-negative.

Dimension: Each X_i is one-dimensional, but actually the X_i may be considered as one observation on an n-dimensional random variable; i.e., the X_i may be dependent.

Moments: $\mu_i=EX_i$

$$\mu=\sum_{i=1}^{n}\mu_i$$

References: Cramér (1946), Fréchet (1937).

Notes:

1. This inequality is formally the same as **1**, but shows how **1** can be used where the random variable of interest is actually the sum of several random variables.

2. It is clear how this inequality is derived from the first one, for it is the same as that one, except that the random variable can be written in two ways, that is, either as X or as a sum of X_i.

2. Bienaymé-Tchebycheff

$$P(|X-\mu|\geq\lambda\sigma)\leq 1/\lambda^2 \tag{2}$$

$$P(X\geq\lambda\sigma+\mu \text{ or } X\leq\mu-\lambda\sigma)\leq 1/\lambda^2 \tag{2'}$$

$$P(|X-\mu|\geq\lambda)\leq\sigma^2/\lambda^2 \tag{2''}$$

Random variable: X

Restrictions: None

Dimension: One

Moments: $\sigma^2=E(X-\mu)^2$

References: Cramér (1946), Fréchet (1937), Uspensky (1937).

Notes:

1. This is the standard Tchebycheff inequality for one random variable.

2. Now the probability is decreasing as $1/\lambda^2$, which means that the probability of large deviations from the mean becomes quite small. It is to be noted that for the normal distribution and for large λ this is actually a very poor estimate of the probability of large deviations, for there the probability of a large deviation is smaller than $e^{-\lambda^2}$, but for intermediate values this is not a bad approximation.

3. If one has a fairly good estimate of σ, then this inequality can be used for testing hypotheses about the mean, and for finding confidence intervals for the mean. In many industrial applications this inequality is used for estimating how much of the production will be near the mean of the process, where one has a good idea of the variance.

2a. Bienaymé-Tchebycheff

$$P(\sqrt{n}|\overline{X}-\mu|\geq\lambda\sigma)\leq 1/\lambda^2 \tag{2a}$$

$$P(|\overline{X}-\mu|\geq\lambda)\leq\sigma^2/n\lambda^2 \tag{2a'}$$

Random variable: $\overline{X}=\frac{1}{n}\sum_{i=1}^{n}X_i$

Restrictions: If $i\neq j$, then X_i and X_j are uncorrelated; that is, $E(X_i-EX_i)(X_j-EX_j)=0$

Dimension: One

Moments: $\mu = EX_i$

$\sigma^2 = E(X_i - \mu)^2$

References: Cramér (1946), Fréchet (1937), Uspensky (1937).

Notes:

1. This is one of the most useful of the Tchebycheff inequalities. One can use this whenever he has sample averages of identically and independently distributed random variables.

2. This inequality gives the "square root of the sample size" law. That is, $|\overline{X} - \mu|$ is of the order of magnitude $1/\sqrt{n}$ in probability.

3. The uses of this inequality are much like those of **2**, but it can be used when working with sample averages.

2b. Bienaymé-Tchebycheff

$$P(|X - \mu| \geq \lambda\sigma) \leq 1/\lambda^2 \tag{2b}$$

$$P(|X - \mu| \geq \lambda) \leq \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ij}/\lambda^2 \tag{2b'}$$

Random variable: $X = \sum_{i=1}^{n} X_i$

Restrictions: None

Dimension: n

Moments: $\mu_i = EX_i$, $\mu = \sum_{i=1}^{n} \mu_i$

$\sigma_{ij} = E(X_i - \mu_i)(X_j - \mu_j)$

$\sigma^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ij}$

Reference: Uspensky (1937).

Notes:

1. This is another form of the Tchebycheff inequality. In this case the random variables forming the sum can have different variances and can be correlated.

2. In the future, inequalities are presented only for samples of one observation. The detailed treatments of the versions of **1** and **2** for sample sums and sample averages can be modified to apply to the other inequalities.

3. Another point of interest is that, although the inequalities appear to require exact estimates of certain moments before they can be used, it is possible to get similar inequalities by substituting upper bounds for the moments that are involved. For instance, in this inequality one might not actually know the value of σ, but from previous experience he might know that it can not under any circumstances be greater than, say, σ'. In this case, if one uses σ' instead of σ, then the inequality will not be as good as the given one, but still may prove useful. This technique can, with some care, be used for all the inequalities.

3. Pearson

$$P\left(\frac{|X - \mu|}{\beta_r^{1/r}} \geq \lambda\right) \leq 1/\lambda^r \tag{3}$$

$$P(|X - \mu| \geq \lambda) \leq \beta_r/\lambda^r \tag{3'}$$

$$P\left(\frac{|X - \mu|}{\sigma} \geq \lambda\right) \leq \frac{\beta_r}{\sigma^r \lambda^r} \tag{3''}$$

Random variable: X

Restrictions: None

Dimension: One

Moments: $\mu = EX$

$\sigma^2 = E(X - \mu)^2$

$\beta_r = E|X - \mu|^r$

References: Fréchet (1937), Narumi (1923), Pearson (1919).

Notes:

1. In order to use this inequality, an absolute moment of the random variable is required.

2. If several absolute moments are available and one needs an inequality for a particular λ, then use that moment that makes the right-hand side of **3'** the smallest for that particular λ. Thus, for instance, **3''** should be compared to **2**: for large values of λ it may provide a smaller bound.

4. Birnbaum, Raymond, and Zuckerman

If n is an even integer,

$$P\left(\sum_{i=1}^{n} (X_i - \mu_i)^2 \geq \lambda^2\right) \leq \begin{cases} 1 & \text{if } \lambda^2 \leq n\sigma^2 \\ \dfrac{n\sigma^2}{2\lambda^2 - n\sigma^2} & \text{if } n\sigma^2 \leq \lambda^2 \leq \dfrac{n\sigma^2}{4}(3+\sqrt{5}) \\ \dfrac{n\sigma^2}{\lambda^2}\left(1 - \dfrac{n\sigma^2}{4\lambda^2}\right) & \text{if } \dfrac{n\sigma^2}{4}(3+\sqrt{5}) \leq \lambda^2 \end{cases}$$

If n is an odd integer,

$$P\left(\sum_{i=1}^{n}(X_i-\mu_i)^2\geq\lambda^2\right)\leq\begin{cases}1 & \text{if } \lambda^2\leq n\sigma^2\\ \dfrac{(n+1)\sigma^2}{2\lambda^2-(n-1)\sigma^2} & \text{if } n\sigma^2\leq\lambda^2\leq\dfrac{\sigma^2}{4}(3n+1+\sqrt{5n^2+6n+5})\\ \dfrac{n\sigma^2}{\lambda^2}-\dfrac{(n^2-1)}{4}\dfrac{\sigma^4}{\lambda^4} & \text{if } \dfrac{\sigma^2}{4}(3n+1+\sqrt{5n^2+6n+5})\leq\lambda^2\end{cases}$$

Random variable: $\sum_{i=1}^{n}(X_i-\mu_i)^2$

Restrictions: The X_i's are independently distributed with common variance.

Dimension: n

Moments: $\mu_i=EX_i$

$$\sigma^2=E(X_i-\mu_i)^2$$

Reference: Birnbaum, Raymond, and Zuckerman (1947).

Notes:

1. This inequality is an upper bound on the probability of the sample point falling outside of a hypersphere centered at the population mean. Birnbaum, Raymond, and Zuckerman (1947) also gives bounds for hyper-ellipses.

2. The application of this inequality to bombing and other aiming problems is obvious.

3. This inequality is multidimensional in that the probability of a multidimensional set is bounded. The random variables, however, are assumed to be independent. Inequality **5** is multidimensional in both senses, e.g., the probability of falling in a rectangle is bounded and the random variables are independent.

4. As n becomes large, the results for odd and even integers n approach each other.

5. Berge

$$P\left(\text{either } \frac{|X_1-\mu_1|}{\sigma_1}\geq\lambda \text{ or } \frac{|X_2-\mu_2|}{\sigma_2}\geq\lambda\right)\leq\frac{1+\sqrt{1-\rho^2}}{\lambda^2} \tag{5}$$

$$P\left(\max\left\{\frac{|X_1-\mu_1|}{\sigma_1},\frac{|X_2-\mu_2|}{\sigma_2}\right\}\geq\lambda\right)\leq\frac{1+\sqrt{1-\rho^2}}{\lambda^2} \tag{5'}$$

$$P\ (|X_1-\mu_1|\geq\lambda_1\sigma_1 \text{ or } |X_2-\mu_2|\geq\lambda_2\sigma_2)\leq\frac{1}{2\lambda_1^2\lambda_2^2}\left[\lambda_1^2+\lambda_2^2+\sqrt{(\lambda_1^2+\lambda_2^2)^2-4\rho^2\lambda_1^2\lambda_2^2}\right] \tag{5''}$$

Random variable: $X=(X_1, X_2)$

Restrictions: None

Dimension: Two

Moments: $\mu_i=EX_i$

$$\sigma_i^2=E(X_i-\mu_i)^2$$

$$\sigma_{12}=E(X_1-\mu_1)(X_2-\mu_2)$$

$$\rho=\frac{\sigma_{12}}{\sigma_1\sigma_2}$$

References: Berge (1938), Lal (1955), Olkin and Pratt (1958), Whittle (1958b).

Notes:

1. This inequality bounds the probability of falling outside of a rectangle centered at the means for a bivariate sample.

2. This inequality uses the dependence between the random variables, and therefore, in order to apply this inequality one needs actually to have some knowledge about the correlation.

3. The right-hand side of this inequality is a decreasing function of the correlation. Thus, the most impressive results are obtained when the correlation is one. Even when the random variables are independent (zero correlation) the right-hand side is $2/\lambda^2$ which is not quite as strong as could be obtained from **2** but still useful.

4. Marshall and Olkin (1960a) found a "one-sided" version of this inequality and its extension to p dimensions.

6. Guttman

$$P\left[(\overline{X}-\mu)^2\geq\frac{S^2}{n-1}+\sigma^2\sqrt{\frac{2(\lambda^2-1)}{n(n-1)}}\right]\leq 1/\lambda^2 \tag{6}$$

$$\lambda\geq 1$$

Random variable: $\overline{X}=\dfrac{\sum_{i=1}^{n}X_i}{n}$

Restrictions: X_i are identically and independently distributed

Dimension: One

Moments: $\mu=EX_i$

$$\sigma^2=E(X_i-\mu)^2$$

and define

$$S^2=\frac{1}{n}\sum_{i=1}^{n}(X_i-\overline{X})^2$$

References: Guttman (1948b), Midzuno (1950).

Note:

Inequality **6** is applicable whenever **2a** can be used, however, **6** takes more computing than **2a** does, and that is its only disadvantage. If we compare these two inequalities, they differ only in the quantity on the right of the inequality sign within the probability statement. For **6** that quantity is a random variable, and it is not for **2a**. The expected size of this random variable is much smaller than that of the quantity that occurs in **2a**, and therefore with probability near one it is a better inequality to use than **2a**.

7. Kolmogorov

$$P(T\geq\sigma\lambda)\leq 1/\lambda^2 \qquad (7)$$

Random variable: $S_i=\sum_{j=1}^{i}(X_j-\mu_j);\ i=1,2,\ldots,n$

$$\text{Let } T=\max_{1\leq i\leq n}[|S_1|,\ldots,|S_i|,\ldots|S_n|]$$

Restrictions: X_i and X_j independent

Dimension: One

Moments: $\mu_i=EX_i$

$$\sigma^2=\sum_{i=1}^{n}E(X_i-\mu_i)^2$$

References: Fréchet (1937), Kolmogoroff (1928), Marshall (1960), Uspensky (1937).

Note:

This inequality can be used whenever **2a** is applicable. A typical use of this inequality is in the extreme-value situations. For instance, if one is putting together an assembly one might ask what is the probability that the cumulative error ever exceeds a certain quantity, and this inequality would give the answer.

8. Cantelli

$$P(X-\mu\leq\lambda)\leq\frac{\sigma^2}{\sigma^2+\lambda^2} \qquad \text{if } \lambda<0 \qquad (8)$$

$$\geq 1-\frac{\sigma^2}{\sigma^2+\lambda^2} \qquad \text{if } \lambda\geq 0$$

Random variable: X

Restrictions: None

Dimension: One

Moments: $\mu=EX$

$$\sigma^2=E(X-\mu)^2$$

References: Cantelli (1928), Cramér (1946), Uspensky (1937).

Notes:

1. This inequality is applicable whenever inequality **2a** can be used.

2. In this case one is interested in one-sided alternatives; that is, one wishes to detect large positive deviations from the mean. This occurs, for instance, whenever one is using one-sided confidence intervals or one-sided test regions.

3. The derivation of this inequality essentially depends on the Schwartz inequality.

4. Comparisons of "one-sided" and "two-sided" alternatives for certain convex sets are obtained by Marshall and Olkin (1960b).

9. Gauss (Camp-Meidell)

$$P(|X-\mu_0|\geq\lambda\tau)\leq\begin{cases}1-\dfrac{\lambda}{\sqrt{3}} & \text{for } \lambda\leq\dfrac{2}{\sqrt{3}}=1.1547\\[2ex] \dfrac{4}{9\lambda^2} & \text{for } \lambda\geq\dfrac{2}{\sqrt{3}}\end{cases} \qquad (9)$$

$$P(|X-\mu|\geq\lambda\sigma)\leq\frac{4}{9}\cdot\frac{1+s^2}{(\lambda-s)^2} \text{ if } \lambda>s \qquad (9')$$

Random variable: X

Restrictions: X has a density function with one mode, μ_0

Dimension: One

Moments: $\mu=EX$

$$\sigma^2=E(X-\mu)^2$$

$$\tau^2=\sigma^2+(\mu-\mu_0)^2$$

$$s=\left|\frac{\mu-\mu_0}{\sigma}\right|$$

References: Cramér (1946), Fréchet (1937), Gauss (1880), Narumi (1923).

Notes:

1. This inequality requires the same knowedge of moments as does inequality **2**; but it is also necessary to know the mode of the distribution. For a symmetric distribution, of course, the mode is the same as the mean.

2. For a symmetric unimodal distribution, this is a better inequality than **2**, since the bound in **2** is here multiplied by 4/9. The inequality **9′** has a particularly simple form when $s=0$. Indeed, for any unimodal distribution such that a bound for s is known, this inequality is better than **2** for sufficiently large values of λ.

3. If this decreasing property actually is true, then this is a better inequality to use than **2**, for it essentially multiplies the bound by 4/9.

4. This inequality in the form **9** is a special case of inequality **10′**.

10. Narumi, Gauss (Camp-Meidell)

$$P(|X-\mu_0|\geq\lambda\sqrt{\nu_r}) \begin{cases} \leq 1-\frac{r+1}{rb}\left(1-\frac{1}{b^r}\right)\lambda \text{ if } 0\leq\lambda\leq\frac{rb}{r+1} \\ \leq\frac{1}{b^r} \text{ if } \frac{rb}{r+1}\leq\lambda\leq b \\ \leq\frac{1}{\lambda^r} \text{ if } b\leq\lambda \end{cases} \quad (10)$$

$$P(|X-\mu_0|\geq\lambda\sqrt{\nu_r}) \begin{cases} \leq 1-\frac{\lambda}{(1+r)^{1/r}} \text{ if } \lambda\leq\frac{r}{(1+r)^{1-1/r}} \\ \leq\left(\frac{r}{r+1}\right)^r(\lambda)^{-r} \text{ if } \frac{r}{(1+r)^{1-1/r}}\leq\lambda \end{cases} \quad (10')$$

Random variable: X

Restrictions: X has a density function $f(x)$, and $f(x)$ has an unique maximum in the interval $(\mu_0-b,\ \mu_0+b)$ at μ_0 and $b>0$. Use (10) if b is finite, and (10′) if b is infinite.

Dimension: One

Moments: $(\nu_r)^r=E|X-\mu_0|^r$

References: Fréchet (1937), Narumi (1923).

Notes:

1. In **10** and **10′** the absolute moments about the mode are used rather than the absolute moments about the mean. For unimodal symmetric distributions that is not a restriction.

2. For many of the common distributions b will be infinity and **10′** can be used in preference to **10**. In applications, however, one might only be sure of the behavior of the density function near the mode and thus **10** is required.

3. Actually, in most cases one would probably use inequality **9**, which is a special case of these inequalities; this is why it has been given by itself.

11. Peek

$$P(|X-\mu|\geq\lambda\sigma)\leq\frac{1-\delta^2}{\lambda^2-2\lambda\delta+1},\ \lambda\geq\delta \quad (11)$$

Random variable: X

Restrictions: None

Dimension: One

Moments: $\mu=EX$
$\sigma^2=E(X-\mu)^2$
$\nu=E|X-\mu|$
$\delta=\nu/\sigma$

Reference: Peek (1933).

Notes:

1. This inequality is much like inequality **2**, except that here one needs to know ν, the mean deviation. If one has this additional information, this may be a better inequality to use.

2. This is a special case of inequality **12**.

11a. Peek

$$P(|X-\mu|\geq\lambda\sigma)\leq\frac{4}{9}\frac{1-\delta^2}{(\lambda-\delta)^2} \quad (11a)$$

Random variable: X

Restrictions: X has a density function whose only mode is its mean

Dimension: One

Moments: $\mu=EX$
$\sigma^2=E(X-\mu)^2$
$\nu=E|X-\mu|$
$\delta=\nu/\sigma$

Reference: Peek (1933).

Note:

This is an improvement over **9** for relatively large λ, but can only be used if one has an estimate of the mean deviation. For a particular value of λ, this should be compared with **9** and with **12**.

12. Cantelli

$$P(|X-\mu|\geq\lambda)\leq\frac{\beta_r}{\lambda^r} \quad \text{if } \lambda^r\leq\frac{\beta_{2r}}{\beta_r}$$

$$\leq\frac{\beta_{2r}-\beta_r^2}{(\lambda^r-\beta_r)^2+\beta_{2r}-\beta_r^2} \quad \text{if } \frac{\beta_{2r}}{\beta_r}\leq\lambda^r \quad (12)$$

Random variable: X

Restrictions: None

Dimension: One

Moments: $\mu=EX$
$\beta_{2r}=E|X-\mu|^{2r}$
$\beta_r=E|X-\mu|^r$

References: Cantelli (1910), Fréchet (1937), Peek (1933).

Notes:

1. Of course in this case one needs information about two moments.

2. The first part of **12** is equivalent to **3′**; for larger values of λ, **12** is better than **3′** when the required two moments are known.

13. Bernstein

$$P(|X-\mu|\geq\lambda)\leq 2e^{-\lambda^2/(2\sigma^2+2C\lambda)} \quad (13)$$

Random variable: $X=\Sigma X_i$

Restrictions: X_i and X_j are independent

Dimension: One

Moments: $\mu_i=EX_i, \qquad \mu=\Sigma\mu_i$

$\sigma_i^2=E(X_i-\mu_i)^2, \qquad \sigma^2=\Sigma\sigma_i^2$

$E|X_i-\mu_i|^s\leq\frac{\sigma_i^2 s!\ C^{s-2}}{2}$ for all integers s and some constant $C(C>0)$.

References: Curtiss (1950), Craig (1933), Fréchet (1937), Uspensky (1937).

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Notes:

1. This is a very nice inequality in the sense that the bound goes to zero very rapidly as λ increases.

2. The one difficulty in applying this inequality is that essentially one has to know at least an upper bound for every moment of each distribution involved. The next inequality treats a useful particular case.

13a. Bernstein

$$P(|X-\mu|\geq\lambda)\leq 2e^{-\lambda^2/(2\sigma^2+\frac{2}{3}m\lambda)} \qquad (13a)$$

Random variable: $X=\Sigma X_i$

Restrictions: $P(|X_i-\mu_i|>m)=0$, X_i and X_j are independent $(i\neq j)$. The maximum deviation of a random variable X_i from its mean μ_i will not exceed m, with probability one.

Dimension: One

Moments; $\mu_i=EX_i$, $\mu=\Sigma\mu_i$

$$\sigma_i^2=E(X_i-\mu_i)^2, \qquad \sigma^2=\Sigma\sigma_i^2$$

References: Fréchet (1937), Uspensky (1937).

Note:

Each X_i has a distribution which does not allow deviations greater than m from μ_i. This condition also bounds all of the central moments.

13b. Bernstein

$$P(|X-\mu|\geq\sigma\lambda)\leq e^{-\lambda^2/2} \qquad (13b)$$

Random variable: $X=\Sigma X_i$

Restrictions: X_i, X_j are independent $(i\neq j)$; X_i is symmetrical about μ_i; i.e., $E(X_i-\mu_i)^{2r+1}=0$ $(r=0, 1, 2, \ldots)$

Dimension: One

Moments: $\mu_i=EX_i$, $\mu=\Sigma\mu_i$

$$E(X_i-\mu_i)^{2r}\leq\left(\frac{\sigma_i^2}{2}\right)^r\frac{(2r)!}{r!} \qquad (r \text{ an integer})$$

$$\sigma_i^2=E(X_i-\mu_i)^2, \ \sigma^2=\Sigma\sigma_i^2$$

Reference: Uspensky (1937).

Note:

This form of the Bernstein inequality places many restrictions on the underlying distributions through their moments. The normal distribution does satisfy these conditions.

III. Tables

Essentially tables 1 and 2 are inverses of each other, column by column. The first table answers the question: How large is the probability associated with a specific "deviation" λ? The second table gives the "deviation" associated with specific probabilities P.

$$P(\overline{X}-\mu\geq\lambda)\leq 0.10,$$

TABLE 1. *Probability associated with deviation* λ

Probability associated with / λ	1	2	3	4	5	6	7
	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$	$\frac{1}{\lambda^4}$	$\frac{4}{9\lambda^2}$	$\left(\frac{4}{5}\right)^4\frac{1}{\lambda^4}$	$\frac{1}{1+\lambda^2}$	$e^{-\lambda^2/2}$
1.0	1.000	1.000	1.000	0.444	0.4096	0.5000	0.607
1.5	0.667	0.444	0.1975+	.198	.0809	.3077	.325
2.0	.500	.250	.0625	.111	.0256	.2000	.135
2.5	.400	.160	.0256	.071	.0105−	.1379	.044
3.0	.333	.111	.0123	.049	.0050	.1000	.011
3.5	.286	.082	.0067	.036	.0027	.0755−	.002
4.0	.250	.062	.0039	.028	.0016	.0588	.000
4.5	.222	.049	.0024	.022	.0010	.0471	↑
5.0	.200	.040	.0016	.018	.0007	.0385−	↑
5.5	.182	.033	.0011	.015−	.0005−	.0320	↑
6.0	.167	.028	.0008	.012	.0003	.0270	↓
6.5	.154	.024	.0006	.011	.0002	.0231	.000

TABLE 2. *Deviation associated with probability* P

Deviation associated / P	1	2	3	4	5	6	7
	$\frac{1}{P}$	$\frac{1}{\sqrt{P}}$	$\frac{1}{P^{2/3}}$	$\frac{2}{3\sqrt{P}}$	$\frac{4}{5P^{2/3}}$	$\sqrt{\frac{1}{P}-1}$	$\sqrt{2\log_e\frac{1}{P}}$
0.99	1.0101	1.0050	1.0025	0.6700	0.802	0.1005−	0.1418
.95	1.0526	1.0260	1.0129	.6840	.810	.2293	.3202
.90	1.1111	1.0541	1.0267	.7027	.821	.3333	.4590
.75	1.3333	1.1547	1.0746	.7698	.860	.5773	.7585
.50	2	1.4142	1.1892	.9428	.951	1	1.1774
.25	4	2	1.4142	1.3333	1.131	1.7321	1.6651
.10	10	3.1623	1.7783	2.1082	1.423	3	2.1460
.05	20	4.4721	2.1147	2.9814	1.692	4.3589	2.4477
.01	100	10	3.1623	6.6667	2.530	9.9499	3.0348
.001	1000	31.6228	5.6234	21.0818	4.500	31.6070	3.7169

An example is now given showing how to use these tables. Suppose one has a sample of nine independent observations from a distribution whose variance is four units squared. (1) What is an upper bound for the probability that the sample average is more than one unit larger than the population mean? (2) How far above the population mean could the ninety percent point of the distribution of sample means be? (3) If one knew the population had an unique mode at its mean, could these results be improved? Answer: For questions (1) and (2) one needs inequality **8**, and for (3) one needs **9′**.

To answer question (1), determine an upper bound for $P(\overline{X}-\mu\geq 1)$ using the second part of **8**, since $\lambda=1>0$. Since **8** is given for a random variable X with variance σ_X^2, and the present question involves the average of $n=9$ observations from a distribution with $\sigma_X^2=4$, we use the variance of $\overline{X}$, $\sigma_{\overline{X}}^2=4/9$, in inequality **8**. Thus,

$$P(\overline{X}-\mu\geq 1)\leq\frac{\sigma_{\overline{X}}^2}{\sigma_{\overline{X}}^2+\lambda^2}=\frac{1}{1+9/4}.$$

The value of this bound is found in column (6) of table 1 for $\lambda=3/2=1.5$, which shows $1/(1+\lambda^2)=0.3077$.

To answer question (2), determine the smallest value of λ for which

again using the second part of **8**. The required value of λ satisfies

$$\frac{\sigma_{\bar{X}}^2}{\sigma_{\bar{X}}^2+\lambda^2}=0.10$$

or

$$\frac{1}{1+\lambda^2/\sigma_{\bar{X}}^2}=0.10.$$

The smallest value of $\lambda/\sigma_{\bar{X}}$ which satisfies inequality **8** is found in column (6) of table 2 for $P=0.10$, which shows $\lambda/\sigma_{\bar{X}}=3$, whence $\lambda=3\sigma_{\bar{X}}=2$.

To answer question (3), the two previous questions are treated again using inequality **9′**. This makes use of the additional information that the distribution is unimodal with its mode μ_0 at the mean ($s=0$ in **9′**). On the other hand, **9′** is a two-sided inequality. The first step is to evaluate an upper bound for

$$P(\bar{X}-\mu\geq 1)\leq P(|\bar{X}\leq\mu|\geq 1)$$

using **9′** with $\lambda\sigma_{\bar{X}}=1$, that is with $\lambda=3/2$. This is found in column (4) of table 1, which shows $4/9\lambda^2=0.198$. This is a smaller upper bound than the one obtained from **8**.

The second part of (3) is to find the smallest value of λ for which

$$P(|\bar{X}\leq\mu|\geq\lambda)\leq 0.10.$$

This value is given by

$$\frac{4}{9}\frac{1}{(\lambda/\sigma_{\bar{X}})^2}=0.10.$$

and from column (4) of table 2 for $P=0.10$ is found to be $\lambda/\sigma_{\bar{X}}=2.1082$ or $\lambda=1.41$. This is better (smaller) than the value obtained by use of **8**.

Thus for this problem it seems better to use inequality **9′** than **8**.

These tables will facilitate choosing which inequality to use when several are available, by comparing the associated probabilities (deviations) with the deviation (probability) of interest, thus making it possible to choose the inequality that gives the smallest probability (deviation) for the problem at hand.

The columns of the tables are associated with the inequalities as follows:

Column	*Inequalities*
1	(**1**), (**1a**), (**1a′**)
2	(**2**), (**2′**), (**2a**), (**2b**), (**6**), (**7**)
3	(**3** with $r=4$)
4	(**9**)
5	(**10′** for $r=4$)
6	(**8** for $\sigma^2=1$)
7	(**13** for $2\sigma^2+2c\lambda=2$; **13a** for $2\sigma^2+\frac{2}{3}m\lambda=2$; **13b**)

IV. Examples

Example (1). Assuming that all soldiers are between 60 and 78 inches tall, what is the probability that the average height of 500 soldiers is more than 1 inch away from the average height of all soldiers?
Solution: Although the population is finite, it is safe to assume that the measurements in the sample are independent. The largest possible variance occurs if half the soldiers have height 60 inches, and half have height 78 inches, in which case the variance is 81 inches squared. First apply inequality **2a′**. Here $\lambda=1$ inch, $\sigma^2=81$ inches squared, and $n=500$. Thus the answer is $\frac{\sigma^2}{n\lambda^2}=\frac{81}{500}=0.162$. One can also apply inequality **13a**. Here $\lambda=500$ inches, $\sigma^2=500\times.81$ inches, and $m=18$ inches. The probability is 0.11, and thus for this example inequality **13** gives more precise results than **2**.

Example (2). In the course of deposit and withdrawal transactions, such as money in a bank, or radioactive material in a hospital, one often wishes to control the absolute error. That is, in a sequence of, say, 100 transactions (a day's activity) one does not want one's books to differ from one's assets, at any time, by more than some fixed amount, say 1,000 units. Assume that the variance due to errors of measuring and of counting for each transaction is 400 units squared (this value being obtained by previous experience). The question then naturally arises: what is an upper bound for the probability of having an accumulated error of more than 1,000 units at any time during the day?
Solution: Inequality **7** is suited for this problem. Here $n=100$, $\sigma=20$, and $\sqrt{n}\sigma\lambda=1{,}000$. Thus $\lambda=\frac{1{,}000}{10\cdot 20}=\frac{1{,}000}{200}=5$, and $1/\lambda^2=1/25=0.04$. If instead of 100 transactions there had been 400, then $n=400$, $\sigma=20$, $\sqrt{n}\sigma\lambda=1{,}000$, $\lambda=\frac{1{,}000}{20\cdot 20}=\frac{1{,}000}{400}=2.5$; and the resulting probabilty is at most $\left(\frac{1}{2.5}\right)^2=\frac{1}{6.25}=0.161$.

Example (3). From previous experience assume that the correlation between two variables (height and weight, rainfall and crop yield) is at least 0.8. If a sample of 25 is made on this bivariate distribution, what is an upper bound for the deviations from the population means that will not be exceeded more than 10 percent of the time, where deviations are measured in standard units?
Solution: Here one can use inequality **5**. First one must solve the following equation:

$$0.10=\frac{1+\sqrt{1-0.64}}{\lambda^2} \text{ or } 0.10=\frac{1+\sqrt{0.36}}{\lambda^2},$$

$$\lambda^2=\frac{1+0.6}{0.1}=\frac{1.6}{0.1}=16;\ \lambda=4.$$

Thus

$$P\left(\frac{|\bar{x}-\mu|}{\sigma_{\bar{x}}}\geq 4 \text{ or } \frac{|\bar{y}-\mu|}{\sigma_{\bar{y}}}\geq 4\right)\leq 0.1,$$

or

$$P\left(\frac{|\bar{x}-\mu|}{\sigma_x}\geq\frac{4}{5} \text{ or } \frac{|\bar{y}-\mu|}{\sigma_y}\leq\frac{4}{5}\right)\leq 0.1.$$

Thus both sample means will be within 0.80 standard units of their respective population means with probability at least 0.9.

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(Paper 65B3–59)

JOURNAL OF RESEARCH of the National Bureau of Standards—B. Mathematics and Mathematical Physics
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Publications of the National Bureau of Standards*

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Selected Abstracts

Torsional resonance vibrations of uniform bars of square cross section, W. E. Tefft and S. Spinner, *J. Research NBS* **65A** (*Phys. and Chem.*) *No. 3, 167* (*May–June 1961*) *75 cents.*

Relations by which the shear modulus may be computed from the fundamental and overtones of the torsional resonance frequencies of square bars have been established empirically.

The results are analyzed in terms of a proportionality factor, R, defined by the equation

$$G=\left(\frac{2lf_n}{n}\right)^2\rho R$$

R is found to increase with increasing cross section to length ratio. Also, the overtones are less than integral multiples of the fundamental by an amount which increases with increasing cross section to length ratio.

Crystallization of bulk polymers with chain folding: theory of growth of lamellar spherulites, J. D. Hoffman and J. I. Lauritizen, Jr. *J. Research NBS* **65A** (*Phys. and Chem.*) *No. 4, 297* (*July–August 1961*) *75 cents.*

A systematic study of the problem of spherulitic growth in linear polymers in bulk has been carried out. A calculation of the radial growth of polymer spherulites is given for four models. These concern growth where the surface nuclei that control the rate are (1) bundlelike and coherent; (2) chain folded and coherent; (3) chain folded and noncoherent; and (4) bundlelike and noncoherent. The required modifications of nucleation theory are given. Then the radial growth rate laws are derived for each model, and the type of "spherulite" that would be formed discussed.

The model with chain folded and coherent growth nuclei leads to a typical lamellar spherulite. The properties of the individual chain folded lamellae that form the spherulite are predicted, including the change of step height with growth temperature, melting behavior, and the behavior on recrystallization. (Chain folded lamellae may also occur in specimens that are not obviously spherulitic.) Under certain conditions, the noncoherent model with chain folds can lead to a modified lamellar spherulite. None of the bundlelike models will lead to a typical lamellar spherulite, though a spherical microcrystalline object might be formed. It is concluded that lamellar spherulites consist largely of chain folded structures.

The factors that could cause chain folded crystals to appear in profusion in bulk polymers are discussed. The case of homogeneous initiation is considered first. Homogeneous initiation of chain folded nuclei in bulk will prevail if the end surface free energy of the bundlelike nucleus exceeds that of the folded. It is shown that the end surface free energy of the bundlelike nucleus, as calculated with a density gradient model, will be larger than had been supposed previously. It is therefore considered to be theoretically possible that the end surface free energy of the bundlelike nucleus may in some cases exceed that of the folded nucleus. Attention is given to the possibility that folded structures appear in large numbers because cumulative strain or large chain ends prevent the growth of bundlelike nuclei to large size, even when the latter type of nucleus is energetically favored when small. Heterogeneous initiation of folded structures is then considered.

Other topics include: (1) conditions that might lead to nonlamellar or nonspherulitic crystallization in bulk; (2) the origin of the twist that is frequently exhibited by the lamellae in spherulites; (3) the transitions that may sometimes occur in the radial growth rate law; and (4) interlamellar links.

Comparison between mode theory and ray theory of VLF propagation, H. Volland, *J. Research NBS* **65D** (*Radio Prop.*) *No. 4, 357* (*July–August 1961*) *75 cents.*

It is shown that the field strength according to mode theory and ray theory in the VLF band are derivable from the same expression of the original vector potential, and the result of one theory is the analytic continuation of the other one in another range of convergence. In fact, both ranges of convergence overlap. Estimates of these ranges are made and an example shows that within this overlapping region (between distances of 300 and 2000 km) both theories give the same result. Using this fact calculations of frequency spectra are possible which in the case of a white noise show some similar features to measured frequency spectra of lightning discharges.

On the validity of some approximations to the Appleton-Hartree formula, K. Davies and G. A. M. King, *J. Research NBS* **65D** (*Radio Prop.*) *No. 4, 323* (*July–August 1961*) *75 cents.*

The validity of some commonly used quasi-transverse and quasi-longitudinal approximations to the Appleton magnetoionic formula is considered. Using the dipole approximation for the earth's magnetic field the various approximations for refractive index are compared with the values computed from the complete formula for various geomagnetic latitudes and a frequency of 2.0 megacycles per second. It is found that certain approximations become very poor only a short distance from where they are exact and so care must be taken in their use. It is shown that a choice of two suitable approximations yields refractive indices of sufficient accuracy for all geomagnetic latitudes. Certain approximations to the group refractive indices are also considered.

The minima of cyclic sums, K. Goldberg, *J. London Math. Soc.* **35,** *262–264* (*1960*).

Given a complex valued function in m variables, defined on a set S, the (average) cyclic sum of this function is defined for $n \geq m$ variables. Letting M_n be the minimum absolute value of this cyclic sum over S, it is proved that $\lim_{n\to\infty} M_n =$ g.l.b. M_n.

An analysis of the accumulated error in a hierarchy of calibrations, E. L. Crow, *IRE Trans. Instrumentation* **I-9** *No. 2, 105–114* (*Sept. 1960*).

Calibrations of many types are performed in a hierarchy of calibration laboratories fanning out from a national standard. Often the statement is made that the accuracy of each echelon of the hierarchy should be 10 times the accuracy of the immediately following echelon. The validity of such statements is examined by deriving formulas for the total error accumulated over the entire sequence when systematic and random errors may occur in each echelon, and by determining how a given total error may be achieved at minimum total cost under reasonable assumptions for the form of the cost-error functions.

Generating functions for formal power series in non-commuting variables, K. Goldberg, *Proc. Am. Math. Soc.* **11,** *No. 6, 988–991* (*Dec. 1960*).

Generating functions in commuting variables are defined for formal power series in non-commuting variables. The effect on these generating functions of transformations on the non-commuting variables is determined. Application is made to the case of $\log f(x)g(y)$.

Tests for regression coefficients when errors are correlated, M. M. Siddiqui, *Ann. Math. Stat.* **31,** *No. 4, 929–938 (Dec. 1960).*

In a previous paper the covariances of least-squares estimates of regression coefficients and the expected value of the estimate of residual variance were investigated when the errors are assumed to be correlated. In this paper we will investigate the distribution of the usual test statistics for regression coefficients under the same assumptions.

On the nature of the crystal field approximation, C. M. Herzfeld and H. Goldberg, *J. Chem. Phys.* **34,** *No. 2, 643–651 (Feb. 1961).*

A new method is developed for the treatment of molecular interactions, and is applied to a system consisting of a hydrogen atom in a 2p state and a hydrogen molecule in the ground state. The interaction of these two species is calculated using ordinary crystal field theory and also the new method. A comparison of the results shows some of the shortcomings of the conventional crystal field theory, and provides corrections to it. The new method consists of (1) expanding all electron terms of the total Hamiltonian for the system which involve interactions between the atom and the ion, thus transforming the interaction Hamiltonian into sums of products of one-electron operators, and (2) of using properly antisymmetrized wave functions made up of products of atom and molecule eigenfunctions. The calculations show the effect of the neglect of overlap and exchange in ordinary crystal field theory.

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Wavelengths and intensities in the first spectrum of bromine, 2000 to 13000 A. J. L. Tech and C. H. Corliss.
Torsional resonance vibrations of uniform bars of square cross section. W. E. Tefft and S. Spinner. (See above abstract.)
Infrared studies of aragonite, calcite, and vaterite type structures in the borates, carbonates, and nitrates. C. E. Weir and E. R. Lippincott.
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Crystallization of bulk polymers with chain folding: theory of growth of lamellar spherulites. J. D. Hoffman and J. I. Lauritzen, Jr. (See above abstract.)
Phase equilibrium relations in the binary system barium oxide-niobium pentoxide. R. S. Roth and J. L. Waring.
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